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Voronoi tessellation for galaxy distribution

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У роботі подано огляд робіт із застосування мозаїки Вороного – одного з найпотужніших методів дослідження властивостей Всесвіту. Зокрема розглянуто використання даного методу для пошуку угруповань у двовимірних і тривимірних розподілах позагалактичних об'єктів, аналізу т.зв. олівцевих оглядів неба, моделювання вибухів наднових зір. Обговорюється обмеження, яке накладає мікрохвильове реліктове випромінювання на використання мозаїки Вороного для дослідження великомасштабного розподілу речовини у Всесвіті.

1. Introduction

The large-scale structure (LSS) of the Universe is formed by galaxies, clusters of galaxies, superclusters of galaxies, and voids. The linear scale of the LSS is varied from 10^{21} cm till 10^{28} cm extending well in space coordinate V ($V = c \times z$, that is *light velocity* $\times$ *redshift*) up to 30 000 km/s.

The first maps of two-dimensional galaxy distribution (projection on celestial sphere) were published by Messier, Abbe, Proctor, Dreyer, Stratonoff, Easton, and others during 1741 – 1922 yrs. For example, Fig. 1 represents the two-dimensional distribution of 11500 brightest galaxies (according to Charlier [1]). This Figure illustrates the real galaxy distribution as well as the so-called zone of avoidance, which is an artifact due to the position of the earth observer in Milky Way (zone of a strong light absorption by interstellar medium of our Galaxy). The observed anisotropy in the distribution of galaxies outside the zone of avoidance is not due to the absorption of light. Since the pioneering papers by Hubble in 1925 – 1929, when the true extragalactic origin of "nebulae" was proved, the origin and spatial distribution of galaxies has become an actual task. Later, due to works by Hubble, Lundmark, Reynolds, Zwicky, Abell, and Kiang, and compilation of the first catalogues of galaxy clusters, this problem has gained in importance: as a result the clustering of galaxies into groups and clusters at various scales has been discovered as well as the conception of galaxy superclusters (e.g. Local Supercluster, de Vaucouleurs, 1953) has been adopted.

During the last two decades an amount of the data on distances to galaxies (or radial velocities of galaxies) has been growing rapidly. The analysis of three-dimensional maps (e.g. the well-known works by Gregory & Thompson (1978); Einasto (1981); Geller & Huchra (1988, 1989)) allows us to conclude that the large-scale galaxy distribution is not random: there are regions with

large concentration of galaxies and those where they are lacking; there is a so-called cellular structure of the LSS. One of the ways to present a spatial galaxy distribution is to compress the data in one spherical coordinate (declination, δ) and to present its as cone diagrams showing the galaxy distribution in another spherical coordinate (right ascension, α) and space coordinate (velocity, V, or redshift, z, of galaxies). In the cone diagram (Fig. 2) we can see the spatial distribution of 1778 galaxies in the region: $\alpha = 8^h - 17^h$; $\delta = 26°.5 - 38°.5$; $V = 0 - 15000$ km/s (according to Geller & Huchra [2]).

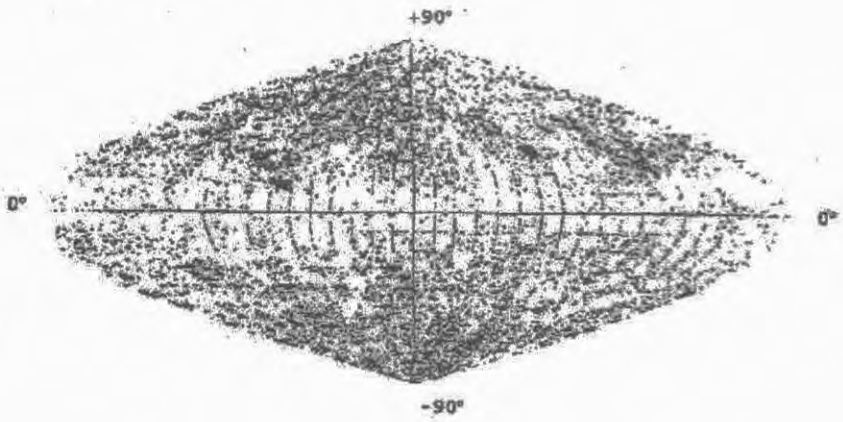

Fig. 1. The observed two-dimensional distribution of 11500 brightest galaxies in the equatorial coordinates α, δ (according to Charlier, 1922, [1]).

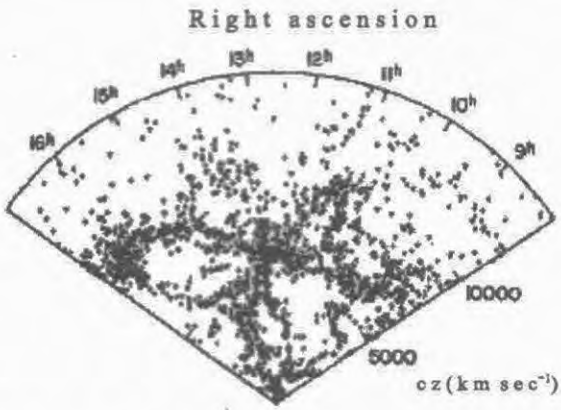

Fig. 2. The cone diagram of the spatial distribution of 1778 galaxies in the region: $\alpha = 8^h - 17^h$; $\delta = 26°.5 - 38°.5$ (the combination of redshift slices processed by groups of the California Astronomers (CfA surveys of galaxies) according to Geller & Huchra, 1989, [2]).

2. Methods for studying the distribution of galaxies

The spatial distribution of galaxies given in Fig. 2 demonstrates that the LSS of galaxies (in this case at the scales up to 15000 km/s) forms a network of cells of the luminous matter (in visible diapason of the electromagnetic spectrum). It consists of galaxies, galaxy clusters, galaxy superclusters, which are localized in walls, filaments, and very dense nodes, and this luminous matter surrounds huge voids. How these observed structures have evolved from the primordial fluctuations of density in the early Universe is the main question for cosmological theories.

Among various geometrical and statistical methods which have been developed and applied for studying global properties of the LSS and for comparison of theoretical models and N-body simulations with observational galaxy distribution, we would like to mention the following: counts of galaxies in cells; mathematical modeling; N-body simulations; n-point correlation functions; cluster method; percolation method; taxonomic method; void probability function; Voronoi tessellation; fractal analysis; wavelet transforms.

The principal limitation of these methods is that they do not take into account real sizes, morphology, masses and other individual properties of galaxies and galaxy clusters regarding them as point objects.

3. Voronoi tessellation

One of the methods allowing us to describe the large-scale galaxy distribution is the Voronoi tessellation [3]. G. Voronoi, an Ukrainian mathematician, proposed this geometrical method in 1908. Later its mathematical technique was improved and detailed by many authors. The idea to apply Voronoi tessellation in cosmological context belongs to Matsuda & Shima [4]. In [6], Icke & Weygart introduced new statistical data to the model of Voronoi tessellation of random Poisson distributions. One of their important conclusions was that the distribution of angles between edges of Voronoi cells is sensitive to the underlying point process. They also noted the useful complementarity of the Voronoi tessellation with N-body techniques, which have a low dynamic range and suffer from sampling noise and boundary effects on very large scales.

Let us consider a Voronoi cell of finite size in N-dimensional space (usually $N = 2$ or $N = 3$), where a fixed number of points is distributed according to some statistical law (for example, the Poisson law). Suppose that each point is the center of a spherical expanding bubble structure. If all these structures begin to expand at the same moment of time with the same rate, the bubbles will be touched in planes that perpendicularly bisect the lines connecting the centers of expansion. These bisecting planes, in turn, intersect each other. As a result of this process, new lines will be generated which in turn intersect each other and form a network. Using an adopted terminology, we will call such a center of the cell as a nucleus. So, each nucleus will be enclosed by a set of

(N - 1) – dimensional planes forming a convex cell. Distribution of nuclei forms the Voronoi tessellation.

Realization of Voronoi tessellations for certain number of expanding nuclei, which is known as the Voronoi foam, can be found in [6, 7]. In the case of two-dimensional realization, the construction of a Voronoi cell consists of the search for all the Delaunay triangles having three nuclei (the center of the circumscribing circle is a vertex of the Voronoi foam). The program proposed by the authors [6], allows one to find all the Delaunay triangles having N_1 as a corner and construct the Voronoi cell belonging to N_1 by joining the circumcentres of the Delaunay triangles. Having applied this procedure to all nuclei, we obtain the Voronoi tessellation. The process is shown in Fig. 3.

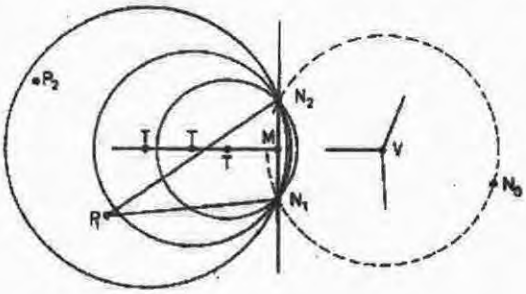

Fig. 3. The construction of a new Delaunay triangle from two known nuclei N_3 such that (N_1, N_2, N_3) forms a triangle whose circumsphere does not contain any other nucleus. The points N_0, N_1, N_2 form a Delaunay triangle obtained in a previous search; corresponding Voronoi vertex V is shown within the (dashed) circumcircle of (N_0, N_1, N_2) as well as stubs of the Voronoi cell walls. On the left hand side of the diagram, the T are a sequence of trial points, the third of which produces a circle that encompasses two nuclei, P_1 and P_2. The radius of the circumcircle of (N_1, N_2, P_1) being smaller than that of (N_1, N_2, P_2), the point P_2 is N_3, i.e. the third corner of the Delaunay triangle. Thus, the circumcenter of (N_1, N_2, P_1) is the next Voronoi vertex which, if connected with V, produces a complete Voronoi cell wall [6].

Algorithm of the Voronoi tessellation is the following: prescribing boundary conditions for the grid where the point distribution of nuclei by some statistical point process is realized, we construct the circumference encompassing all points that lie closer to the given nucleus than to any other. We can also correct initial parameters of tessellations. As a result we will have a number of characteristics describing Voronoi cells, which allows us to compare tessellations with the real observed distribution.

These characteristics for each cell are the circumference, area, number of vertices, length of each wall, angle between each pair of neighboring walls,

distance between the center and each vertex, perpendicular distance between the center and each wall (Fig. 4). According to [6], the only three moments are known analytically for a two-dimensional Voronoi cell (in the case of a homogeneous Poisson point distribution): the number of vertices (E(N) = 6), the area (E(A) = 1/q), the perimeter (E(S) = $4/\sqrt{q}$). The only six moments are known analytically for a three-dimensional Voronoi polyhedron (in the case of a homogeneous Poisson point distribution): the volume (E(V) = 1/q), the surface area (E(S) = $[256\pi/3]^{1/3}\Gamma(5/3)q^{-2/3}$), the edge length (E($L_1$) = $[4^{5/3}\pi^{5/3}\Gamma(1/3)]/[3^{5/3}5q^{1/3}]$), the number of edges (E(N_1) = $144\pi^2/35$), the number of faces (E(N_2) = $[48\pi^2/35] + 2$), the number of sections per unit area cut by a plane (E(M_1) = $[4^{2/3}\pi^{5/2}\Gamma(1/3)]/[3^{5/3}5q^{1/3}]$). Here q is the number of points per unit volume in N-space.

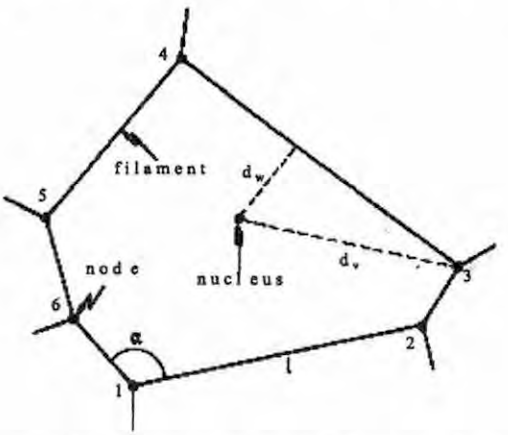

Fig. 4. Identification of the four quantities which were calculated in each Voronoi cell: l_i, the length of wall i; α, the angle between two walls meeting at vertex; d_w, the distance between the nucleus and a wall (note that the projection of the nucleus doesn't necessarily lie on the wall) [6].

4. Voronoi tessellation as a tool for studying 2-D galaxy distribution

It has been shown [5] that the regions of lower density in the early Universe become more and more spherical with time. Thus the large scale morphology of the high density baryonic material in the Universe, consisting of "clusters" in the form of walls, filaments, and nodes can be obtained when matter streams away from the distributed low-density expansion centers ("nuclei") and collects in the interstices of close packing spheres [6]. In kinematical calculation [7], the density in the centers of walls became lowest quite quickly during the time of evolution of the walls; latter, the density in the filaments becomes lowest

and streams to nodes. So, the walls became "poorest". In addition to super dense fields of matter (superclusters and clusters of galaxies), the existence of voids is observed.

Applying the Voronoi tessellation for describing the global properties of the LSS of galaxies (Fig. 2), we can find and evaluate such structure elements as one-dimensional filaments, two-dimensional walls, and three-dimensional nodes within observational surveys of galaxies. In [6, 7], the authors have given a description of the LSS in terms of the Voronoi tessellation and obtained statistical properties of such structures. They also have pointed out that low-density domains (voids) play a dominant role in the formation of the LSS.

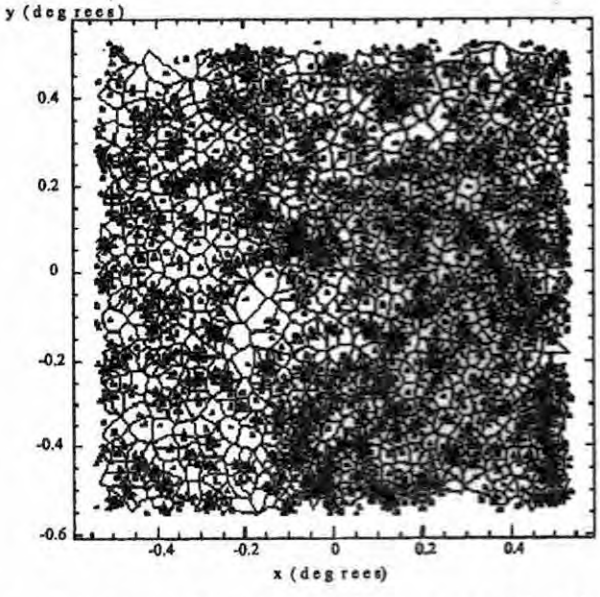

Fig. 5. Voronoi tessellation of a galaxy field generated by the Shewchuk "triangle" C code [8].

The automated procedure for detecting clusters of galaxies within a two-dimensional photometric catalog of galaxies is presented in the work of Ramella *et al.* (2002) [8]. The authors developed the Voronoi Galaxy Cluster Finder (VGCF), which uses galaxy position and magnitudes to find clusters and determine their size, richness and contrast above background. In contrast to the matched filter method, kernel based techniques, and wavelet transforms, the VGCF is sensitive both to symmetric and elongate and/or irregular clusters. This property is very important since a large fraction of clusters has the elongated structure. In Fig. 5 we can see the Voronoi tessellation of a galaxy field obtained in [8], when the "triangle" C code by Shewchuk (http://www.cs.cmu. edu/~quake/triangle.html) was realized as the algorithm. For finding the over densities in an observational galaxy survey, the number of galaxies per area unit is a more interesting parameter than the area of a Voronoi cell.

So, establishing the density threshold for significant over densities at the 80 % level, the authors identified 37 clusters on the Palomar Distant Cluster Survey (PDCS). Of these clusters, 12 are VGCF counterparts of the 13 PDCS clusters (z = 0.2 - 0.6) with 3σ confidence level; 2 are PDCS clusters with confidence level $< 3\sigma$; some fraction of the remaining 23 systems detected by VGCF may have missed by the matched filter algorithm.

5. Voronoi tessellation as a tool for studying 3-D galaxy distribution and pencil-beam surveys

The three-dimensional Voronoi tessellation consists of four structure elements: the walls surrounding the polyhedral cells, the edges, and the nodes. The extension to 3D is following: four nuclei form a Delaunay tetrahedron if it has a circumsphere that has no other nuclei inside. The center of the circumsphere is the Voronoi vertex [5]. It has been shown [6, 7] that a Voronoi foam in three-dimensional space may be constructed on the base of three topological elements: walls associated with planes which bound cells (voids); filaments (edges) are the places where three walls are crossing each other; nodes are the points of crossing by four filaments. Using the terminology of the LSS, the planes can be identified with walls in the galaxy distribution (see Fig. 2 and the "Great Wall" structure as the example), filaments correspond to the elongated superclusters, nodes are identified with rich clusters of galaxies.

Icke and Weygaert have argued [9] that the Voronoi 3D tessellation can be applied to the gravitational instability models with the structure formation dominated by the negative density fluctuations; to the models with high-frequency cutoff in fluctuation spectra or considerable power at low wavelength; to the models where the driving force is due to explosions (like the Supernova explosion).

Doroshkevich *et al.* (see, for example, [10]) applied the core-sampling method to various Voronoi tessellations. It has allowed them to identify one- and two dimensional structure elements (filaments and walls), to evaluate the surface density of filaments and the linear density of walls, and as a result to demonstrate an efficiency of the core-sampling method. The Voronoi tessellations according to the Weygaert code were used for the creation of synthetic samples.

In 1990, a striking quasi-periodic galaxy redshift distribution was found in a pencil-beam (one-dimensional) survey (Fig. 6). The mean spacing between peaks is equal to $128h^{-1}$ Mpc (1 Mpc = 3×10^{18} cm). Both the quasi-periodicity, and its observable scale can be explained by the Voronoi model. It was shown that even in the totally random Voronoi foam with the Poisson distribution of nuclei with mean density n = $(100h^{-1}Mpc)^{-3}$, the mean spacing is equal to $137h^{-1}$ Mpc. Simulations of deep magnitude-limited pencil-beam surveys (see references in [9]) have shown that 15 % of the beams show the periodicity of the order 105 - 150 h^{-1} Mpc out to depth of z = 0.5. Some statistics along

simulated pencil-beams are given in [11]. The exact expression for the distribution of spacing between peaks in a three-dimensional Voronoi tessellation is obtained in [12].

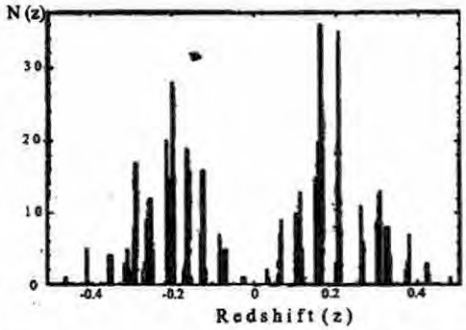

Fig. 6. A deep pencil beam redshift survey through a Voronoi foam ($m_{1,m} = 21.5$). The redshift distribution for pencil beam surveys taken in opposite directions through a 3-D Voronoi tessellation. Positive redshift means the galaxy is in the cone pointing the one direction, while negative redshift means the opposite direction [9].

6. Voronoi model of the LSS and microwave background constraints

The cosmic microwave background radiation (T = 2°.7 K), which is the relict radiation formed in the recombination epoch of the early Universe is one of the main arguments for our correct understanding of the LSS formation. "Whether it is possible for the Universe to emerge from the radiation epoch with a network of under-densities such as to create a connected Voronoi network by gravitational expansion alone?" Resolving this problem, Coles and Barrow showed in [13] that Voronoi LSS models, in which primordial under-densities expand faster than the mean Hubble flow produce large microwave background temperature anisotropies over angular scales up to one degree. Observational limits on anisotropies ($dT/T \sim 10^{-3}$) on such scales constrain the formation of a galaxy cluster (at very low redshifts) unless the initial conditions are highly non-Gaussian. By other words, "the amplitudes of under-densities required to create a percolated tessellation network are so large that they violate existing bounds on the isotropy of the microwave background". It means that there is a significant re-ionization of the Universe or the primordial under-densities develop from explosive, non-gravitational, origins. This result is sustainable and gives some constraints on the use of the Voronoi model to the LSS distribution.

7. Voronoi tessellation for explosion scenario

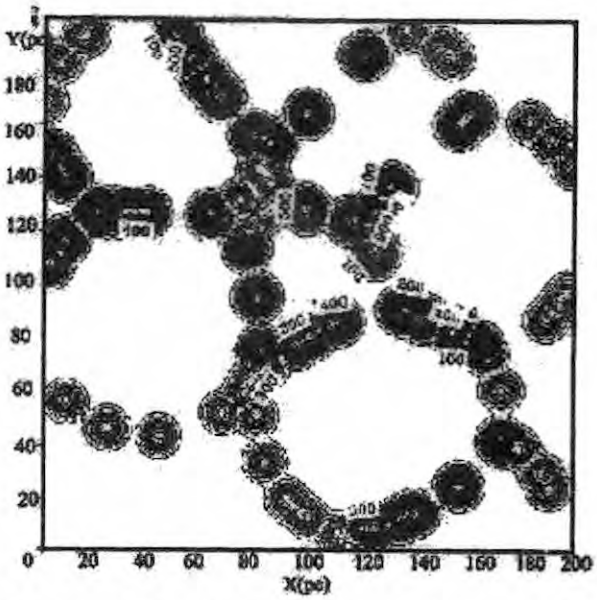

Fig. 7. Emissive contours after Supernova explosion from the tessellation (according to [14]).

Application of the Voronoi tessellation to the explosion scenario likely Supernova was demonstrated by Zaninetti for two-dimensional [14] and three-dimensional case [15]. Since these results are presented by the author in this volume (pp. ...), we would only like to ascertain a conclusive remark. Constructing the Voronoi model on the base of latticed square, the author developed a dynamical method which allows describing the explosion phases of Supernova to simulate the emissive radio/optical contours (Fig. 7), as well as to explain luminous arcs in clusters.

8. Conclusion

The application of the Voronoi tessellation is not confined with the aforementioned examples. Besides the LSS and explosion Supernova scenario, it is very useful for studying the quasar distribution, search of cellular structure in the distribution of Lyman- α clouds, galaxy image analysis by adaptive optics, etc. An elegant geometry allowing one to derive easy statistical properties with high statistical significance makes the Voronoi tessellation an efficient tool for extragalactic research [16].

References

[1] C.V.L. Charlier (1922), *How an infinitive world may be built up*, Arkiv for Matematik, astronomi och Fysik **16**, 22.

[2] M.J. Geller, J.P. Huchra (1989), *Mapping the Universe*, Science **246**, 897-903.

[3] G. Voronoï (1908) *Nouvelles applications des parameters continus a la theorie des formes quadratques. Deuxieme Memorie: recherches sur les paralleloedres primitives*, J. reine angew. Math. **134**, 198-246.

[4] T. Matsuda, E. Shima (1984), *Topology of Supercluster-Void Structure*, Progress of Theoretical Physics **71**, 855-858.

[5] J. Centrella, A.L. Melott (1983), *Three dimensional simulation of large-scale structure in the Universe*, Nature **305**, 196–198.

[6] V. Icke, R. van de Weygaert (1987), *Fragmenting the Universe I. Statistics of two-dimensional Voronoi foams*, Astron. Astrophys. **184**, 16–32.

[7] R. van de Weygaert, V. Icke (1989), *Fragmenting the Universe* II. *Voronoi vertices as Abell clusters*, Astron. Astrophys. **213**, 1–9.

[8] M. Ramella, W. Boschin, D. Fadda, M. Nonino (2001), *Finding galaxy clusters using Voronoi tesselations*, Astron. Astrophys. **368**, 776–786.

[9] V. Icke, R. van de Weygaert (1991), *The galaxy distribution as a Voronoi foam*, Q.J.R. astr. Soc. **32**, 85–112.

[10] A. Doroskevich, S. Gottlober, S. Madsen (1997), *The accuracy of parameters determined with the core-sampling method: Application to Voronoi tessellations*, Astron. Astrophys. Suppl. Ser. **123**, 495–506.

[11] M.U. Subba Rao, A.S. Szalay (1992), *Statistics of pencil beams in Voronoi foams*, Astrophys. J. **391**, 483–493.

[12] S. Ickeuchi, E.L. Turner (1991), *Quasi-periodic structures in the large-scale galaxy distribution and three-dimensional Voronoi tessellation*, Mon. Not. R. astr. Soc. **250**, 519–522.

[13] P. Coles, J.D. Barrow (1990), *Microvawe background constraines on the Voronoi model of large-scale structure*, Mon. Not. R. astr. Soc. **244**, 557–562.

[14] I. Zaninetti (1989), *Dynamical Voronoi tessellation*. I, Astron. Astrophys. **224**, 345–350.

[15] I. Zaninetti (1990), *Dynamical Voronoi tessellation*. II, Astron. Astrophys. **233**, 293–300.

[16] O. Melnyk, A. Elyiv, I.Vavilova (2005), Kinemat.Fiz.Neb.Tel (submitted).

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