https://doi.org/10.15407/knit2022.04.018
UDC 629.78

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ANALYTICAL MODEL OF SATELLITE MOTION IN ALMOST CIRCULAR ORBITS UNDER THE INFLUENCE OF ZONAL HARMONICS OF GEOPOTENTIAL

The article deals with the movement of satellites in low near-circular orbits of the Earth. An analytical model is constructed, which consists of formulas describing the change of the osculating elements and averaged equations. An algorithm for constructing a second approximation of the influence of zonal harmonics of the geopotential on the movement of satellites in almost circular orbits is presented. For the second and third zonal harmonics, formulas are given for the osculating and average elements describing the motion of the satellite in the second approximation in small parameters. The introduction of special variables for almost circular orbits made it possible to significantly simplify the procedure for constructing the second approximation of the influence of zonal harmonics. The article provides a justification for the accuracy of the analytical model for the considered orbits. The constructed model of changes in the average elements of the orbit describes the basic principles of motion. With a sufficiently high accuracy, this model describes the changes in the average elements of the orbit with simple analytical formulas and is convenient for analyzing the properties of orbits and pre-selecting a reference orbit for a specific mission.

Keywords: *analytical model, almost circular orbits, zonal harmonics, average elements, laws of motion.*

INTRODUCTION

The article deals with the movement of satellites in low and very low, almost circular orbits of the Earth. The focus is on orbits with altitudes from 400 to 800 km, although the results obtained can be extended to other low orbits. An almost circular orbit is understood as an orbit for which the changes in radius during one revolution of orbital motion do not exceed tenths of a percent. It is assumed that the inclination of the orbits is not small. Such a choice of orbits is determined by the interest in commercial, fairly light Earth remote sensing (ERS) satellites.

The choice of a reference orbit — an idealized trajectory in the vicinity of which the satellite will move is a necessary and important task of effective satellite mission planning. The requirements for the reference

Цитування: Pirozhenko A. V., Maslova A. I., Vasyliev V. V. Analytical model of satellite motion in almost circular orbits under the influence of zonal harmonics of geopotential. *Space Science and Technology*. 2022. **28**, № 4 (137). P. 18—30. https://doi.org/10.15407/knit2022.04.018

orbit are, of course, contradictory: it should also give high accuracy in predicting movement and be simple enough to allow for its effective selection. It is clear that the rational choice of the reference orbit should be based on knowledge of the basic principles of the perturbed motion of the satellite.

Methods of numerical integration of equations of motion, the use of which is very effective in many problems of dynamics, are ineffective in the task of determining the basic principles of motion. Therefore, from the very beginning of space exploration, analytical theories of the motion of satellites in near-Earth orbits have been created [2, 11]. Work on the development of analytical theories of satellite motion has never stopped (see, for example, [1, 3—6, 8, 17]).

The Simplified General Perturbation (SGP) model series occupies a special position among the analytical models of satellite motion. These models were developed by U.S. Air Force for the purpose of operational monitoring of changes in near-Earth space. The SGP models use average orbital elements, whose values are obtained using a special procedure for removing short-period variations [10]. SGP models have been developed over decades [17] and are now widely used [7, 15].

The difference between the gravitational field of the Earth and the Newtonian central field has the main disturbing effect on the movement of the satellite in low Earth orbits. The main difference between the Earth's gravitational field and the Newtonian one is described by low-order zonal harmonics. At the same time, the effect on the satellite of disturbances from these harmonics does not depend on the rotation of the Earth. Thus, the study of the influence of zonal harmonics on satellite motion makes it possible to make significant progress in determining the basic principles of satellite motion, whilst appearing to be one of the simplest tasks in the study of the influence of external disturbances on satellite motion.

To date, many effective studies have been devoted to the influence of zonal harmonics of the geopotential on the motion of the satellite (see, for example, [1—6, 8, 11, 17]). Therefore, the question arises: why do we need another study? The answer to this question consists of several parts. Firstly, another study will not significantly change the total number of such studies. And the use of new variables [13, 14] describing the motion of the satellite will allow us to see the regularities of motion from a slightly different angle, which can expand our knowledge. Secondly, considering a rather narrow class of orbits together with the special variables introduced for this class made it possible, it seems, to significantly simplify the procedure for constructing a second approximation of the influence of zonal harmonics. Thirdly, the analytical models proposed in the article describe the motion of the satellite depending on the argument of the latitude of the orbit. This, in contrast to models using the mean anomaly or a combination of it, in some cases, seems more convenient. Fourthly, in the available publications on the issue under consideration, it is difficult to analyze the relationship between the so-called mean elements, briefly describing the basic principles of motion and the osculating elements of the orbit. For example, in the monograph [17], a lot of attention is paid to this problem. But among the extensive and very practical general tips for solving this problem, the following phrases are particularly memorable: "Everything has to be consistent!" and "Unfortunately, you won't always find this level of detail." In [16], in the same connection, it is noted that "...the older analytical theories often do not deliver the required accuracy, and the implementation of the newer theories requires access to the internal documentation of other space agencies or journal papers of limited access..." From this, we can conclude that the solution to the problem of the connection between the mean elements and the osculating elements of the orbit requires additional development.

Thus, the construction of an analytical model of the influence of zonal harmonics on the movement of satellites in low Earth orbits is an important task for designing the orbits of remote sensing satellites, and its solution requires additional research. The article proposes an algorithm for constructing an analytical model and formulas for osculating and mean elements describing the motion of the satellite in the second approximation in small parameters under the influence of zonal harmonics of the geopotential.

PROBLEM STATEMENT

The potential of the Earth's gravitational field can be described using a series expansion in spherical functions

$$U=\frac{\mu}{R}\left[1+\sum_{n=2}^{\infty}\left(\frac{R_E}{R}\right)^n\left(C_{n0}P_n(\sin\delta)+\right.\right.$$
$$\left.\left.+\sum_{m=1}^{n}P_{nm}(\sin\delta)(C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)\right)\right], \quad (1)$$

where μ is gravitational constant of the Earth; R is the distance from the center of the Earth to the considered point in space with geocentric latitude δ and longitude λ in the coordinate system associated with the Earth; R_E is the average equatorial radius of the Earth; C_{n0}, C_{nm}, S_{nm} are dimensionless coefficients depending on the distribution of the Earth's masses; $P_n(\sin\delta)$ are Legendre polynomials of order n; $P_{nm}(\sin\delta)$ are associated Legendre functions of order n and index m.

The members of expression (1) containing $P_n(\sin\delta)$ are called the second, third, etc. zonal harmonics, and the terms containing $P_{nm}(\sin\delta)$ are sectorial (at $n=m$) and tesseral (at $0<m<n$) harmonics. The geometrical meaning harmonics is detailed in [17].

It is known that the main changes in the motion parameters of satellites in low orbits are caused by the influence of zonal harmonics. Moreover, harmonics of a lower order have a more significant influence, the magnitude of which is determined by the coefficients C_{n0}: $C_{20}\approx-1.0826\cdot10^{-3}$, $C_{30}\approx2.5324\cdot10^{-6}$, $C_{40}\approx1.6199\cdot10^{-6}$, $C_{50}\approx2.2775\cdot10^{-7}$ [17].

To describe the motion, we will use a special form of equations for close to circular orbits [13, 14]. This form of equations describes the deviation of the satellite trajectory from the circular unperturbed orbit. For this, dimensionless variables b_1, b_2, γ are introduced, associated with the current position and speed of the satellite by the relations

$$R=R_0(1+b_1),\ \dot{R}=b_2\sqrt{\mu/R_0},\ p=R_0(1+\gamma),$$

where R_0 is the radius of an undisturbed circular comparison orbit, p is the focal parameter of the satellite orbit.

The satellite motion equations can be written as

$$i'=z\cos u F_n^*,\quad \Omega'=z\frac{\sin u}{\sin i}F_n^*,$$
$$\Delta u'=\left(\frac{s^{1/2}}{z^2}-1\right)-\Omega'\cos i,\quad b_1'=b_2, \quad (2)$$
$$b_2'=\frac{\gamma-b_1}{z^3}+F_r^*,\qquad \gamma'=2zsF_\tau^*,$$

where the prime denotes the derivative with respect to $\tilde{u}$, $\tilde{u}$ is the argument of the latitude of the unperturbed orbit,

$$\dot{\tilde{u}}=\sqrt{\frac{\mu}{R_0^3}};$$

i, Ω, u are the inclination, the longitude of the ascending node, and the argument of the latitude of the satellite's orbit, respectively; $z=1+b_1$ is the dimensionless radius of the orbit equal to the ratio of the radius of the orbit to the comparison orbit radius, $s=1+\gamma$ is the dimensionless focal parameter of the orbit equal to the ratio of the focal parameter of the orbit to the focal parameter of the comparison orbit;

$$\Delta u=u-\tilde{u};\ F_r^*=\frac{R_0^2}{\mu}F_r,\ F_{\tau,n}^*=\frac{R_0^2}{\mu}s^{-1/2}F_{\tau,n},\ F_r, F_\tau, F_n$$

are radial, transversal, and normal accelerations respectively;

$$\frac{\mu}{R_0^2}$$

is the acceleration of free fall for R_0.

The accelerations from the second zonal harmonic are three orders of magnitude higher than the rest of the disturbing accelerations. To describe its influence, we introduce a small parameter

$$\varepsilon=-\frac{3}{2}C_{20}\frac{R_E^2}{R_0^2}$$

(for R_0 = 7000 km, $\varepsilon=1.35\cdot10^{-3}$). We will consider motion in orbits close to circular in the sense that the initial values of the parameters b_1, b_2, γ are small quantities, the order of which is equal to or greater than the order of smallness of the quantity ε.

The task is to construct an analytical model with reasonable (required for the orbits under consideration) accuracy describing the change in satellite motion under the influence of zonal harmonics of geopotential.

DETERMINATION OF THE REQUIREMENTS FOR THE ACCURACY OF THE ANALYTICAL MODEL

The qualitative requirement for the analytical model in its description consists of the basic principles of satellite motion: the model should describe secular and long-period movements. The model should also allow analyzing the properties of orbits, in particular, their stability. It is desirable that the model includes fairly simple ratios that allow the choice of the satellite's reference orbit.

The constructed first approximation of the influence of the second zonal harmonic [14] shows that it does not allow one to judge the stability of the shape of the orbits. At the same time, it is known that the analysis of the influence of the second and third zonal harmonics makes it possible to determine stable, so-called frozen orbits. The analysis of the influence of higher-order zonal harmonics is incorrect without constructing a second approximation of the influence of the second zonal harmonic. Thus, the study of the regularities of the influence of zonal harmonics and the analysis of the stability of the orbital shapes requires the construction of a second approximation of the influence of the second zonal harmonic of the geopotential.

Consider the quantitative requirements for the accuracy of the analytical model. The first approximation of the solution of the equations of satellite motion with respect to a small parameter ε has an error of the order of $10^{-6}u$, where u is the argument of latitude. We will assume that the accuracy is determined by values an order of magnitude lower, i.e., in the case under consideration, the accuracy is about $10^{-5}u$. Consequently, the expected accuracy of the first approximation in the interval of two orbital loops will be of the order of 10^{-4}, and in the interval of 20 orbits — of the order of 10^{-3}. Taking into account that the radius of the considered orbits is less than 7000 km, we find that the expected accuracy of the first approximation is about 700 m for two orbits and 7 km for 20 orbits. This accuracy does not seem to be sufficient.

To determine the sufficient accuracy of analytical approximations of the influence of zonal harmonics on the satellite's motion, let us consider estimates of the influence of various perturbations on the satellite's motion in the considered orbits (Figure 1, Table 1).

The second approximation from the second zonal harmonic has an error of the order of $\varepsilon^3 u \approx 10^{-9} u$.

Table 1. **Analysis of the influence of various disturbing factors on the satellite motion [12]**

Component of accelerations	Order of magnitude of disturbances (m/s^2)	
	Orbits of altitude of 19000...20000 km (GLONASS, GPS)	Low orbits of altitude 350...400 km (ISS)
Central field of the Earth	0.61	8.8
Effect of Earth flattening (harmonic 2 × 0)	10^{-4}	2.5×10^{-2}
Effect of harmonics of an order higher than 2 × 0	2×10^{-7}	10^{-5}
Effect of harmonics of an order higher than 8 × 8	10^{-10}	4×10^{-7}
Effect of harmonics of an order higher than 36 × 36	0	10^{-7}
Effect of harmonics of an order higher than 72 × 72	0	10^{-8}
Earth's atmosphere	0	10^{-6}
Lunar gravity	4×10^{-6}	10^{-6}
Displacement of Earth's pole from Z-axis of the Geocentric Coordinate System (GCS)	10^{-6}	3×10^{-7}
Solar gravity	10^{-6}	2.5×10^{-7}
Forces of light pressure from the Sun	10^{-7} (GPS)	6×10^{-8} (ISS)
Precession and nutation of Earth's axis of rotation	2.5×10^{-8}	6×10^{-8}
Gravitational disturbances caused by the change of the Earth's shape due to tidal effects on the Earth of the Moon and the Sun	2×10^{-9}	1.5×10^{-7}
Nonuniformity of Earth rotation	3×10^{-9}	7×10^{-9}
Change of Earth's shape due to pole displacement	10^{-11}	2×10^{-9}
Forces of light pressure from the Earth	1.5×10^{-9} (GPS)	4×10^{-9} (ISS)
Forces caused by light and heat radiation of SC	1.4×10^{-9}	10^{-9}
Gravitational disturbances from Venus	1.1×10^{-10}	3×10^{-11}

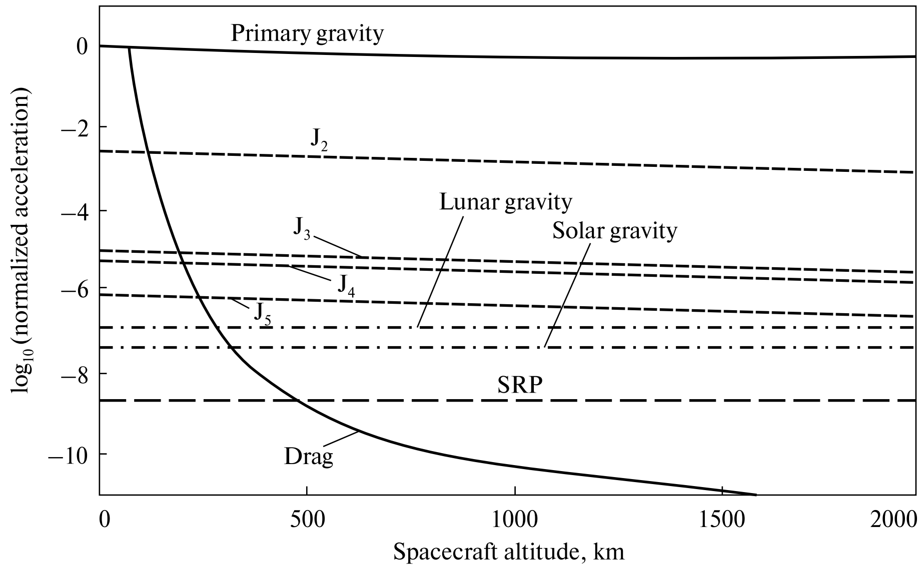

Figure 1. Comparison of various disturbing accelerations in low Earth orbits [9]

Taking into account that the coefficients of the remaining zonal harmonics are at least three orders of magnitude smaller than the coefficient of the second zonal harmonic (i.e., they have the order of the square of the coefficient of the second zonal harmonic and higher), the first approximation of their influence will have an error of a higher order of smallness than the second approximation from the second zonal harmonic. Consequently, the accuracy of the analytical model, including the second approximation from the second zonal harmonic and the first approximations from the remaining zonal harmonics, will be of the order of $10^{-8}u$.

The influence of the rest of the disturbing factors on the satellite's motion in the considered orbits can be estimated by the formula

$$\frac{W}{g_0}u,$$

where W is the acceleration of the satellite due to this effect, and

$$g_0 = \frac{\mu}{R_0^2}$$

is the acceleration of gravity for a given altitude. From the third column of Table 1, it is easy to see that the effects of aerodynamics and lunar gravity are an order of magnitude superior to the accuracy of the second approximation of the influence of the second zonal harmonic, and many other factors have an effect comparable to the accuracy of the second approximation.

The accuracy achieved when constructing the second approximation of the effect of the second zonal harmonic on the satellite's motion $10^{-8}u$ with an orbit radius of less than 7000 km ensures the accuracy of estimating the satellite's position on two turns of approximately 0.7 m and 7 m on twenty turns. Calculations of the motions of satellites and their comparison with measurements of these motions show that such accuracy of the analytical assessment of the influence of zonal harmonics is quite sufficient (the effect of tidal effects has comparable changes in the motions of satellites in low orbits).

Thus, the construction of the second approximation of the influence of the second zonal harmonic and the first approximations of the influence of the higher-order zonal harmonics provides sufficient accuracy for assessing these influences. Constructing approximations of a higher order has no practical meaning because of the effects of other disturbing forces.

CONSTRUCTION OF THE SECOND APPROXIMATION OF THE INFLUENCE OF THE SECOND ZONAL HARMONIC

The disturbing accelerations from the action of the second zonal harmonic have the form

$$F_r = -\frac{3C_{20}\mu R_E^2}{2R^4}(3\sin^2 u \sin^2 i - 1),$$

$$F_\tau = \frac{3C_{20}\mu R_E^2}{2R^4}\sin 2u \sin^2 i, \qquad (3)$$

$$F_n = \frac{3C_{20}\mu R_E^2}{2R^4}\sin u \sin 2i .$$

Passing in (2) to differentiation with respect to *u*, we obtain

$$i' = zw\cos u F_n^*, \quad \Omega' = zw\frac{\sin u}{\sin i}F_n^*,$$

$$\Delta u' = w\left(\frac{s^{1/2}}{z^2} - 1\right) - \Omega'\cos i, \quad b_1' = wb_2, \qquad (4)$$

$$b_2' = w\frac{\gamma - b_1}{z^3} + wF_r^*, \qquad \gamma' = 2wzsF_\tau^*$$

where, as in (2)

$$F_{\tau,n}^* = \frac{R_0^2}{\mu}s^{-1/2}F_{\tau,n}, \quad F_r^* = \frac{R_0^2}{\mu}F_r,$$

$$w = \left(\frac{s^{1/2}}{z^2} - z\,\mathrm{ctg}\,i\sin u F_n^*\right)^{-1},$$

but the prime denotes the derivative with respect to *u*.

Substituting accelerations (3) into (4) we obtain

$$i' = -\frac{\varepsilon}{2}\frac{w}{z^3 s^{1/2}}\sin 2u \sin 2i,$$

$$\Omega' = -2\varepsilon\frac{w}{z^3 s^{1/2}}\cos i \sin^2 u,$$

$$\Delta u' = w\left(\frac{s^{1/2}}{z^2} - 1\right) - \Omega'\cos i, \qquad (5)$$

$$\gamma' = -2\varepsilon w\frac{s^{1/2}}{z^3}\sin^2 i \sin 2u,$$

$$b_1' = wb_2,$$

$$b_2' = w\frac{\gamma - b_1}{z^3} + \varepsilon\frac{w}{z^4}(3\sin^2 i \sin^2 u - 1),$$

where

$$w = \left(\frac{s^{1/2}}{z^2} + 2\varepsilon\frac{1}{s^{1/2}z^3}\cos^2 i \sin^2 u\right)^{-1}.$$

Since we are considering an almost circular orbit and b_1, b_2, γ are small quantities, then, taking into account that with the preservation of only terms of the first order of smallness $w \approx 1 - 0.5\gamma + 2b_1 - 2\varepsilon\cos^2 i \sin^2 u$, in the first approximation the equations with differentiation with respect to *u* coincide with the equations for differentiation with respect to $\tilde{u}$.

We will assume that at the initial moment of time $u = \tilde{u} = 0$, i.e. the trajectory “starts” at the ascending node of the orbit. Then the solutions of equations (5) for i, Ω, γ in the first approximation with respect to small parameters have the form [14]

$$i = i_0 + \Delta i_2 = i_0 + \frac{\varepsilon}{4}\sin 2i_0(\cos 2u - 1),$$

$$\Omega = \Omega_0 + \Delta\Omega_2 = \Omega_0 - \frac{\varepsilon}{2}\cos i_0\left(2u - \sin 2u\right),$$

$$\gamma = \gamma_0 + \Delta\gamma_2 = \gamma_0 + \varepsilon\sin^2 i_0(\cos 2u - 1), \qquad (6)$$

where the subscript “0” denotes the initial values of the variables, and the subscript “2” denotes the terms describing, in the first approximation, changes in motion under the influence of the second zonal harmonic.

We write the equation for the change of b_1 in the form [14]

$$b_1'' + b_1 = -\frac{\varepsilon}{2}\sin^2 i_0 \cos 2u + \gamma_0 + \varepsilon\left(\frac{1}{2}\sin^2 i_0 - 1\right). \qquad (7)$$

Taking

$$\gamma_0 = \varepsilon\left(1 - \frac{1}{2}\sin^2 i_0\right), \qquad (8)$$

we obtain that equation (7) describes harmonic oscillations relative to the zero position. Condition (8) is not a restriction on the values of the focal parameter (transverse satellite velocity) due to the absence of restrictions on the radius of the comparison orbit R_0.

Then the equation describing the change in b_1 takes the form

$$b_1'' + b_1 = -\frac{\varepsilon}{2}\sin^2 i_0 \cos 2u.$$

We write its solution in the form

$$b_1 = b_0\cos u + b_0'\sin u + \frac{d}{3}(\cos 2u - \cos u) =$$

$$= A_0\cos(u - \alpha_0) + \frac{d}{3}(\cos 2u - \cos u),$$

where

$$d = \frac{\varepsilon}{2}\sin^2 i_0 ;$$

A_0, α_0 are the amplitude and phase shift of natural oscillations, $d/3$ is the amplitude of forced oscillations.

Let us construct a solution to equations (5) in the second approximation in small parameters. To do this, we introduce new variables: $i = i_l + i_{sq}$, $\gamma = \gamma_l + \gamma_{sq}$, $\Omega = \Omega_l + \Omega_{sq}$, where the indices "*l*" and "*sq*" denote the components of the solution of equations (5), proportional to the first and second degrees of smallness, respectively.

To describe the changes in b_1, b_2, we introduce new variables A, α as follows

$$b_1 = A\cos(u-\alpha) + \frac{d}{3}(\cos 2u - \cos u),$$

$$b_2 = -A\sin(u-\alpha) + \frac{d}{3}(\sin u - 2\sin 2u).$$

Then the changes in A, α, are described by the equations

$$A' = -\frac{d}{3}[2\cos 2u \sin(u-\alpha) - 2\sin(u+\alpha) + \sin\alpha] +$$
$$+ b_{1r}\cos(u-\alpha) - b_{2r}\sin(u-\alpha),$$

$$A\alpha' = A + \frac{d}{3}[2\cos 2u \cos(u-\alpha) + 2\cos(u+\alpha) - \cos\alpha] +$$
$$+ b_{1r}\sin(u-\alpha) + b_{2r}\cos(u-\alpha),$$

where b_{1r}, b_{2r} are the right-hand sides of the corresponding equations (5).

Since the second approximation is being sought, it is sufficient to express w in the first approximation. Substituting solutions (6) and (8) for γ into the expression for w, we obtain

$$w \approx 1 + 2b_1 - 0.5\varepsilon(3 - 3.5\sin^{\cdot} i) +$$
$$+ \varepsilon(1 - 1.5\sin^2 i)\cos 2u .$$

We substitute the introduced variables into equations (5) and transform them, discarding the terms of the equations of the third and higher order of smallness. After rather cumbersome, but not complicated transformations, it is possible to obtain expressions for the terms Δx_{22}, where $x = \{i, \Omega, \gamma, A, \alpha\}$, describing the change in the parameters of the orbit under the action of the second zonal harmonic of the geopotential in the second approximation. These expressions are given in Appendix.

We note that to construct approximations for A and α, it is more convenient to use the equations in the following form

$$A' = b_{1r}\cos(u-\alpha) - b_{2r}\sin(u-\alpha) -$$
$$-b_2\cos(u-\alpha) - b_1\sin(u-\alpha) - d\cos 2u \sin(u-\alpha), \quad (9)$$

$$A\alpha' = b_{1r}\sin(u-\alpha) + b_{2r}\cos(u-\alpha) -$$
$$-b_2\sin(u-\alpha) + b_1\cos(u-\alpha) + d\cos 2u \cos(u-\alpha). \quad (10)$$

Note also that the cumbersomeness of the formulas, especially for A and α, makes it desirable to verify them. To carry out such a check, differential equations were written out in the most general form, with only linear and quadratic terms preserved on their right-hand side. The verification of the approximation formulas was carried out by comparing these formulas with the results of the numerical integration of the obtained differential equations.

CONSTRUCTION OF THE FIRST APPROXIMATION OF THE INFLUENCE OF THE THIRD ZONAL HARMONIC

Since the zonal harmonics coefficients C_{n0} at $n > 2$ have an order of smallness equal to the square of the second zonal harmonic coefficient and higher, then, for the considered model accuracy, it is sufficient to take into account their influence only in the first approximation. Taking into account the algorithm for constructing a second approximation of the influence of the second zonal harmonic, it is not difficult to understand that the influence of higher-order zonal harmonics will be described by additional terms that can be obtained independently of the influence of other zonal harmonics. Then, taking into account the influence of n zonal harmonics, the orbital parameters will be described as follows

$$x = x_0 + \Delta x_2 + \Delta x_{22} + \Delta x_3 + \ldots + \Delta x_n , \quad (11)$$

where $x = \{i, \Omega, \gamma, A, \alpha\}$ is the orbital parameter; x_0 is its initial value; Δx_2 are the terms describing the influence of the second zonal harmonic in the first approximation; Δx_{22} are the terms describing the influence of the second zonal harmonic in the second approximation; Δx_3 are the terms describing the influence of the third zonal harmonic in the first approximation, etc.

Taking into account the limited volume of the article, and the uniformity of the procedures, we present formulas only for the third zonal harmonic.

The disturbing accelerations of the third zonal harmonic have the form

$$F_r = -2\frac{C_{30}\mu R_E^3}{R^5}(5\sin^2 u \sin^2 i - 3)\sin u \sin i,$$

$$F_\tau = \frac{C_{30}\mu R_E^3}{2R^5}(15\sin^2 u \sin^2 i - 3)\cos u \sin i,$$

$$F_n = \frac{C_{30}\mu R_E^3}{2R^5}(15\sin^2 u \sin^2 i - 3)\cos i.$$

Substituting these accelerations into equations (4) and linearizing the equations, it is easy to obtain formulas for additional terms describing the influence of the third zonal harmonic

$$\Delta i_3 = \frac{1}{2}\varepsilon_3 \cos i_0 \sin u(5\sin^2 i_0 \sin^2 u - 3),$$

$$\Delta\Omega_3 = 1\frac{1}{2}\varepsilon_3 \mathrm{ctg} i_0 \times$$

$$\times\left[5\sin^2 i_0\left(\frac{2}{3} - \cos u + \frac{1}{3}\cos^3 u\right) + \cos u - 1\right],$$

$$\Delta\gamma_3 = \varepsilon_3 \sin i_0 \sin u(5\sin^2 i_0 \sin^2 u - 3),$$

$$\Delta A_3 = \varepsilon_3 \sin i_0 \left\{\frac{3}{4}\sin\alpha_0\left(1 - \frac{5}{8}\sin^2 i_0\right) + \right.$$

$$-\frac{3}{4}\left(1\frac{1}{4}\sin^2 i_0 - 1\right)\sin(2u - \alpha_0) -$$

$$\left. -\frac{5}{16}\sin^2 i_0\left[\sin(2u + \alpha_0) - \frac{1}{2}\sin(4u - \alpha_0)\right]\right\},$$

$$\Delta A_3 = \varepsilon_3 \sin i_0 \left\{\frac{3}{4}\sin\alpha_0\left(1 - \frac{5}{8}\sin^2 i_0\right) + \right.$$

$$+1\frac{1}{2}\left(1\frac{1}{4}\sin^2 i_0 - 1\right)\cos\alpha_0 \cdot u -$$

$$-\frac{3}{4}\left(1\frac{1}{4}\sin^2 i_0 - 1\right)\sin(2u - \alpha_0) -$$

$$\left. -\frac{5}{16}\sin^2 i_0\left[\sin(2u + \alpha_0) - \frac{1}{2}\sin(4u - \alpha_0)\right]\right\},$$

$$\Delta\alpha_3 = \frac{\varepsilon_3}{A_0}\sin i_0 \left\{\frac{3}{4}\cos\alpha_0\left(1 - \frac{5}{8}\sin^2 i_0\right) - \right.$$

$$-1\frac{1}{2}\left(1\frac{1}{4}\sin^2 i_0 - 1\right)\sin\alpha_0 \cdot u +$$

$$+\frac{3}{4}\left(1\frac{1}{4}\sin^2 i_0 - 1\right)\cos(2u - \alpha_0) -$$

$$\left. -\frac{5}{16}\sin^2 i_0\left[\cos(2u + \alpha_0) + \frac{1}{2}\cos(4u - \alpha_0)\right]\right\}.$$

Here, as before, we assume that at the initial moment of time $u = 0$,

$$\varepsilon_3 = \frac{C_{30}R_E^3}{R_0^3}$$

formulas for A and α are obtained using equations (9), (10).

Note that the obtained expressions show that the effect of the third zonal harmonic leads to a systemic change only in the shape of the orbit (α and A). The rest of the parameters are subject to only periodic fluctuations.

NUMERICAL ESTIMATES OF THE MODEL'S ACCURACY

Numerical integration of the equations of orbital motion confirms the above estimates of the accuracy of the analytical formulas. Figure 2 shows the difference $\Delta x = x_{num} - x_{an}$, where $x = \{i, \Omega, \gamma, A, \alpha\}$; x_{num} is the value of the parameter obtained by numerical integration of the equations of orbital motion; x_{an} is the value of the parameter obtained using the constructed analytical approximations in accordance with formula (11). The calculations were carried out taking into account the effect of the second and third zonal harmonics for the following initial conditions:

$$R_0 = R_{sr} + 507 \text{ km},\ R_{sr} = 6371 \text{ km},\ i_0 = 97.4^\circ,$$

$$\Omega_0 = 183.3^\circ,\ b_{10} = -4.8\cdot 10^{-4},\ b_{20} = -0.00126$$

$$(A_0 \approx 0.00135,\ \alpha_0 \approx 249.12^\circ),\ u_0 = 0.$$

AVERAGED EQUATIONS

We obtain the averaged equations by applying the averaging operator

$$\frac{1}{2\pi}\int_0^{2\pi} f(u)du$$

to the right-hand sides of differential equations (4), taking into account (9), (10), in which terms not higher than the second order of smallness are preserved.

The equations averaged over u, taking into account the second and third harmonics, have the form

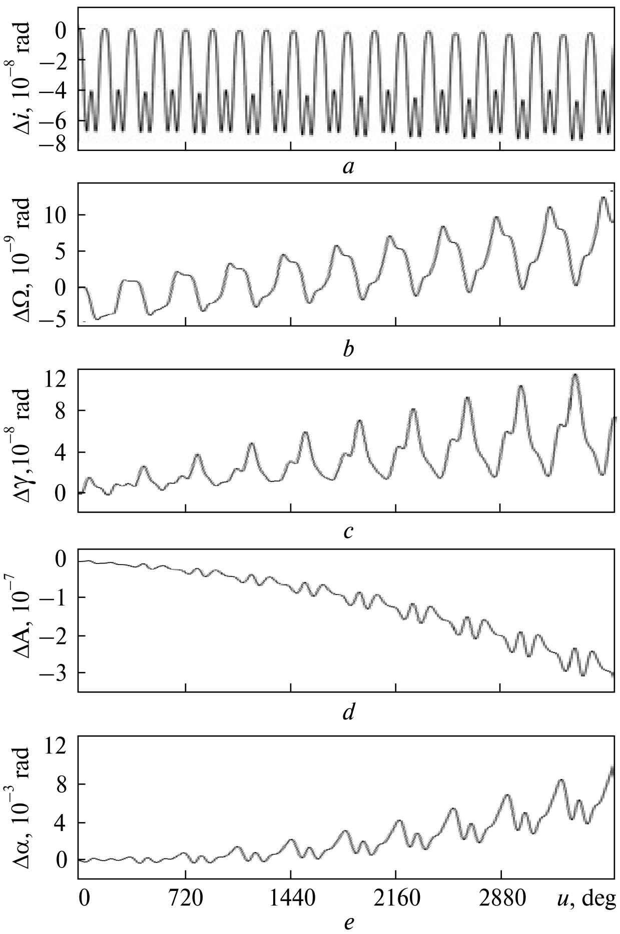

Figure 2. Difference in inclination (*a*), in longitude of the ascending node (*b*), in the description of the relative focal parameter γ (*c*), in the description of the amplitude of the change in the relative radius of the orbit (*d*), in the description of the phase of oscillations of the orbital radius (apogee argument) (*e*)

$$\bar{i}' = 0,\ \ \bar{\gamma}' = 0,$$

$$\bar{\Omega}' = -\varepsilon\cos\bar{i} + \varepsilon\cos\bar{i}\left(2\frac{1}{2}\varepsilon - 8\frac{2}{3}d\right),$$

$$\bar{A}\bar{\alpha}' = -C\sin\bar{\alpha} - \bar{A}G + \frac{d}{3}G\cos\bar{\alpha}, \qquad (12)$$

$$\bar{A}' = C\cos\bar{\alpha} + \frac{d}{3}G\sin\bar{\alpha},$$

where the "hat" denotes the average values;

$$G = 5d - 2\varepsilon;$$

$$C = 1\frac{1}{2}\varepsilon_3\sin\bar{i}\left(1\frac{1}{4}\sin^2\bar{i} - 1\right).$$

It is clear that the right-hand sides of equations (12) fully correspond to the linear terms of the previously constructed approximations.

Note that the use of averaged equations is correct when G has order ε and C has order ε^2. The case when G and C have higher orders of smallness ε^2 and ε^3, respectively, requires additional research. Further, orbits for which $\sin^2 i_0 \approx 0.8$ are not considered.

To use the averaged equations in constructing a long-term forecast of satellite movements, it is necessary to determine the initial conditions for the averaged equations. Unfortunately, the averaging method does not allow this to be done. Calculations show that accepting the initial conditions equal to the initial conditions of the initial equations leads to significant errors. The combination of the averaging method with the constructed analytical approximations makes it easy to solve this problem. In accordance with the logic of the averaging method, we require that high-frequency oscillations be carried out relative to the average solution with zero mean. Then the initial conditions for the averaged equations are determined by the free terms of the expansions. For example, the initial value for $\bar{i}$ is

$$\bar{i}_0 = i_0 - \frac{\varepsilon}{4}\sin 2i_0 +$$

$$+\varepsilon\sin 2i_0\left\{\frac{1}{48}\left(29\varepsilon - 84\frac{1}{3}d\right) + \frac{1}{3}A_0\cos\alpha_0\right\}.$$

The solutions of the averaged equations constructed with such initial conditions show good agreement with the solutions of the original equations. So, the difference between Ω and $\bar{\Omega}$ for the same initial conditions as in the construction of Figures 2 does not exceed $2.5\cdot 10^{-4}$ deg. per 1000 satellite orbital turns. Under the same initial conditions, but $i_0 = 45^\circ$, this difference does not exceed $2\cdot 10^{-3}$ deg. for 1000 turns. Note that the deviations between $\bar{\Omega}$ and mean Ω grow linearly with time. This growth is apparently associated with unaccounted for accelerations from the second zonal harmonic (see Figure 2, *b*). The linear nature of the deviations of $\bar{\Omega}$ from the mean Ω allows them to be reduced, if this is required by the task.

Figure 3 shows the changes in the amplitudes $A, \bar{A}$ and phase shifts (apogee arguments) $\alpha, \bar{\alpha}$ of the initial (2) and averaged equations for 1000 satellite orbits. The initial conditions of motion are the same as for Figures 2. The solutions of the complete equations are shown in the figure by the solid line; the solutions of the averaged equations are shown by the circles. The difference between α and $\bar{\alpha}$ for 1000 turns is less than 0.2°, between A and $\bar{A}$ less than $2.5 \cdot 10^{-6}$.

Equations (12) are easy to integrate. Let us construct a solution for $\bar{A}$ and $\bar{\alpha}$. We introduce new variables $\lambda = \bar{A}\cos\bar{\alpha}$, $h = \bar{A}\sin\bar{\alpha}$. The change in these variables is described by the equations

$$\lambda' = \bar{A}'\cos\bar{\alpha} - \bar{A}\bar{\alpha}'\sin\alpha =$$

$$= \bar{A}G\sin\bar{\alpha} + C = Gh + C,$$

$$h' = \bar{A}'\sin\bar{\alpha} + \bar{A}\bar{\alpha}'\cos\bar{\alpha} =$$

$$= \frac{d}{3}G - \bar{A}G\cos\bar{\alpha} = \frac{d}{3}G - G\lambda.$$

Or compactly

$$\lambda' = Gh + C,$$

$$h' = \frac{d}{3}G - G\lambda. \quad (13)$$

The solution of equations (13) for $G > 0$ and $G < 0$ can be written in the form

$$\bar{A}\sin\bar{\alpha} = -\left(\bar{A}_0\cos\bar{\alpha}_0 - \frac{d}{3}\right)\sin Gu +$$

$$+\left(\bar{A}_0\sin\bar{\alpha}_0 + \frac{C}{G}\right)\cos Gu - \frac{C}{G},$$

$$\bar{A}\cos\bar{\alpha} = \frac{d}{3} + \left(\bar{A}_0\cos\bar{\alpha}_0 - \frac{d}{3}\right)\cos Gu + \quad (14)$$

$$+\left(\bar{A}_0\sin\bar{\alpha}_0 + \frac{C}{G}\right)\sin Gu,$$

where $\bar{A}_0, \bar{\alpha}_0$ are the initial conditions of the averaged equations.

It follows from (14) that motion has a unique equilibrium position, which is determined by the equalities

$$\bar{A}_0\sin\bar{\alpha}_0 = -\frac{C}{G}, \ \bar{A}_0\cos\bar{\alpha}_0 = \frac{d}{3}.$$

And this equilibrium position is stable.

Thus, the averaged equations describe the basic principles of satellite motion by simple ratios. Having a sufficiently high accuracy, these equations allow analyzing the properties of the orbits and making a preliminary choice of the parameters of the reference orbit for the satellite mission.

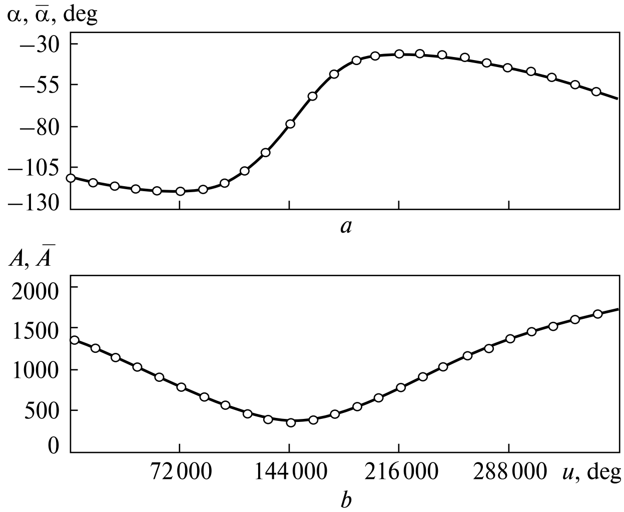

Figure 3. Solutions of the initial (2) and averaged equations for the amplitude and phase shift in the oscillations of the orbit radius

The combination of solutions of the averaged equations and formulas of the second approximation makes it possible to construct relations for the long-term prediction of satellite motions. For this, in the obtained formulas of the second approximation, it is necessary to replace the free and linear terms with the corresponding solutions of the averaged equations, and replace the initial values in the formulas of the second approximation with "initial" values that correspond to the current values of the average elements.

CONCLUSIONS

1. The constructed analytical model of the second approximation in small parameters describes with sufficient accuracy short-period changes in the motion of satellites in low, almost circular orbits under the influence of the second and third zonal harmonics of the geopotential.

2. The proposed algorithm for constructing the second approximation of the influence of the zonal harmonics of the geopotential on the motion of satellites in almost circular orbits, despite the cumbersome formulas, is mathematically simple. It consists of schemes for constructing the second approximation of the influence of the second zonal harmonic

and the first approximations of the influence of higher order zonal harmonics. The algorithm allows one to easily take into account the influence of any number of zonal harmonics.

3. The constructed model of changes in the average elements of the orbit describes the basic principles of motion. Having a sufficiently high accuracy, the model describes the changes in the average elements with simple analytical formulas. The proposed model is convenient for analyzing the properties of orbits and the preliminary selection of a reference orbit for a specific mission.

APPENDIX

$$\Delta i_{22} = \varepsilon \sin 2i_0 \left[\frac{1}{48}\left(29\varepsilon - 84\frac{1}{3}d\right) + \frac{1}{3}A_0\cos\alpha_0 - \right.$$

$$-\frac{1}{4}A_0\cos(u+\alpha_0) + \frac{1}{12}d\cos u + \left(2d - \frac{11}{16}\varepsilon\right)\cos 2u +$$

$$+\frac{1}{36}d\cos 3u - \frac{1}{12}A_0\cos(3u-\alpha_0) +$$

$$\left. +\frac{1}{16}\left(\varepsilon - 4\frac{1}{3}d\right)\cos 4u + \frac{1}{48}(\varepsilon - 4d)\cos 6u \right],$$

$$\Delta\Omega_{22} = \varepsilon\cos i_0 \left[1\frac{1}{3}A_0\sin\alpha_0 + \left(2\frac{1}{2}\varepsilon - 8\frac{2}{3}d\right)u - \right.$$

$$-\frac{1}{6}d\sin u + A_0\sin(u-\alpha_0) - \frac{1}{2}A_0\sin(u-\alpha_0) +$$

$$+\left(5\frac{2}{3}d - 1\frac{1}{2}\varepsilon\right)\sin 2u + \frac{1}{18}d\sin 3u -$$

$$\left. -\frac{1}{6}A\sin(3u-\alpha_0) + \left(\frac{1}{8}\varepsilon - \frac{2}{3}d\right)\sin 4u \right],$$

$$\Delta\gamma_{22} = 2\frac{2}{3}A_0 d\cos\alpha_0 + 3d\varepsilon - 6\frac{13}{18}d^2 + \frac{2}{3}d^2\cos u -$$

$$-2A_0 d\cos(u+\alpha_0) + 4d(2d-\varepsilon)\cos 2u + \frac{2}{9}d^2\cos 3u -$$

$$-\frac{2}{3}A_0 d\cos(3u-\alpha_0) + d\left(\varepsilon - 2\frac{1}{6}d\right)\cos 4u,$$

$$\Delta A_{22} = -A_0 d - \left(\frac{3}{4}A_0^2 - 4\frac{7}{36}d\varepsilon + 9\frac{7}{12}d^2\right)\cos\alpha_0 -$$

$$-\frac{1}{4}A_0\left(\varepsilon + 2\frac{1}{3}d\right)\cos 2\alpha_0 - \frac{1}{4}A_0^2\cos 3\alpha_0 +$$

$$+d\sin\alpha_0\left(1\frac{2}{3}d - \frac{2}{3}\varepsilon\right)u + A_0 d\cos u +$$

$$+\left(\frac{3}{4}A_0^2 - 2\frac{1}{6}d\varepsilon + 5\frac{1}{18}d^2\right)\cos(u-\alpha_0) +$$

$$+\left(6\frac{1}{36}d^2 - 2\frac{2}{3}d\varepsilon\right)\cos(u+\alpha_0) + 1\frac{1}{6}A_0 d\cos(u-2\alpha_0) +$$

$$+\frac{1}{4}A_0(\varepsilon - 3d)\cos(2u-2\alpha_0) - \frac{1}{18}d^2\cos(2u-\alpha_0) +$$

$$+\frac{1}{12}d\left(3\frac{2}{3}d - \varepsilon\right)\cos(2u+\alpha_0) +$$

$$+\frac{1}{9}d\left(5\varepsilon - 10\frac{3}{4}d\right)\cos(3u-\alpha_0) +$$

$$+\frac{1}{2}d\left(\frac{1}{2}\varepsilon - 1\frac{4}{9}d\right)\cos(3u+\alpha_0) -$$

$$-\frac{1}{6}A_0 d\cos(3u-2\alpha_0) + \frac{1}{4}A_0^2\cos(3u-3\alpha_0) -$$

$$-\frac{1}{9}d^2\cos(4u-\alpha_0) + \frac{1}{3}A_0 d\cos(4u-2\alpha_0) +$$

$$+\frac{1}{6}d\left(1\frac{2}{3}d - \frac{1}{2}\varepsilon\right)\cos(5u-\alpha_0),$$

$$\Delta\alpha_{22} = \left[\frac{1}{4}A_0 + \frac{d}{A_0}\left(9\frac{7}{12}d - 4\frac{7}{36}\varepsilon\right)\right]\sin\alpha_0 +$$

$$+\frac{1}{4}\left(\varepsilon + 2\frac{1}{3}d\right)\sin 2\alpha_0 + \frac{1}{4}A_0\sin 3\alpha_0 +$$

$$+\left[2\varepsilon - 5d + \frac{1}{3}\frac{d}{A_0}\cos\alpha_0(5d - 2\varepsilon)\right]u -$$

$$-\frac{d}{3}\left(1 + \frac{2}{3}\frac{d}{A_0}\cos\alpha_0\right)\sin u +$$

$$+\left[\frac{1}{4}A_0 + 3\frac{1}{3}d\cos\alpha_0 + \frac{d}{A_0}\left(5\frac{1}{6}d - 2\frac{1}{6}\varepsilon\right)\right]\sin(u-\alpha_0) +$$

$$+\frac{1}{3}\frac{d}{A_0}\left(8\varepsilon - 17\frac{3}{4}d\right)\sin(u+\alpha_0) - \frac{1}{2}d\sin(u-2\alpha_0) +$$

$$+\left[-\frac{1}{2}\varepsilon+1\frac{5}{6}d+\frac{d}{A_0}\cos\alpha_0\left(-\frac{17}{18}d+\frac{1}{6}\varepsilon\right)\right]\sin 2u+$$

$$+\frac{1}{12}\frac{d}{A_0}(5d-\varepsilon)\sin(2u-\alpha_0)+\frac{1}{6}\frac{d^2}{A_0^2}\sin(2u+\alpha_0)+$$

$$+\frac{1}{4}(\varepsilon-3d)\sin(2u-2\alpha_0)+$$

$$+\frac{d}{A_0}\left(-1\frac{7}{36}d+\frac{5}{9}\varepsilon\right)\sin(3u-\alpha_0)+$$

$$+\frac{d}{A_0}\left(\frac{13}{18}d-\frac{1}{4}\varepsilon\right)\sin(3u+\alpha_0)-$$

$$-\frac{1}{6}d\sin(3u-2\alpha_0)+\frac{1}{4}A_0\sin(3u-3\alpha_0)-$$

$$-\frac{1}{9}\frac{d^2}{A_0}\sin(4u-\alpha_0)+\frac{1}{3}d\sin(4u-2\alpha_0)+$$

$$+\frac{d}{A_0}\left(\frac{5}{18}d-\frac{1}{12}\varepsilon\right)\sin(5u-\alpha_0).$$

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Стаття надійшла до редакції 17.01.2022
Після доопрацювання 28.04.2022
Прийнято до друку 02.05.2022

Received 17.01.2022
Revised 28.04.2022
Accepted 02.05.2022

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АНАЛІТИЧНА МОДЕЛЬ РУХУ СУПУТНИКА НА МАЙЖЕ КОЛОВИХ ОРБІТАХ ПІД ВПЛИВОМ ЗОНАЛЬНИХ ГАРМОНІК ГЕОПОТЕНЦІАЛУ

Розглядається рух супутників на низьких майже колових орбітах Землі. Побудовано аналітичну модель, яка складається з формул, що описують зміну оскулюючих елементів орбіти, та осереднених рівнянь. Наведено алгоритм побудови другого наближення впливу зональних гармонік геопотенціалу на рух супутників по майже колових орбітах. Для другої та третьої зональних гармонік наведено формули для оскулюючих та середніх елементів, що описують рух супутника у другому наближенні за малими параметрами. Введення спеціальних змінних для майже колових орбіт дозволило значно спростити процедуру побудови другого наближення впливу зональних гармонік. Дано обґрунтування точності аналітичної моделі для аналізованих орбіт. Побудована модель зміни середніх елементів орбіти описує основні закономірності руху. Маючи досить високу точність, ця модель описує зміни середніх елементів орбіти простими аналітичними формулами і зручна для аналізу властивостей орбіт та попереднього вибору опорної орбіти для конкретної місії.

Ключові слова: аналітична модель, майже колові орбіти, зональні гармоніки, середні елементи, закономірності руху.