ESTIMATIONS OF THE ENERGY QUASI-INTEGRAL OF THE RESTRICTED THREE-BODY PROBLEM

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We consider the restricted three-body problem for a dust particle in the vicinity of a spherical cometary nucleus in an eccentric orbit about the Sun. The differential equations of the particle's spatial motion are integrated both analytically and numerically to obtain and estimate the energy quasi-integral.

INTRODUCTION

The three-dimensional generalization of a dust motion in the cometary orbital plane is a problem of natural interest, because it enables one to study macroscopic volume formations in a cometary atmosphere [3, 4, 7–10].

Dust particles coming out of the nuclear region are being acted upon by the forces of radiation pressure and gravitation. The resulting action depends upon the ratio of these two forces, which is generally denoted as β [1, 5, 11]. For the particles with $\beta = 1$, one may term these dust particles *resonant*. In the equation of motion of a resonant particle, gravitational effect of the cometary nucleus will be remained as well beyond a region of the nuclear influence on the non-resonant dust. Because of this, for the resonant particle, one has the three-body problem, that under known conditions can be reduced to the restricted three-body problem, in which the orbit of comet is conic section of arbitrary eccentricity and the trajectory of the dust particle is a spatial curve [6]. In this paper we consider some consequences of the specified eccentric restricted problem in its general case.

DIFFERENTIAL EQUATIONS OF MOTION

The differential equations of motion of a separate dust particle in a non-inertial cometocentric reference system (CRS) are as follows:

$$
\begin{aligned}
\ddot{x}'_1 &= -\mu_{\mathrm{s}}(1-\beta)\frac{r + x'_1}{y^3} - \mu_{\mathrm{c}}\frac{x'_1}{x^3} + \ddot{\theta}x'_2 + \dot{\theta}^2 x'_1 + 2\dot{\theta}\dot{x}'_2 + \frac{\mu_{\mathrm{s}}}{r^2},\\
\ddot{x}'_2 &= -\mu_{\mathrm{s}}(1-\beta)\frac{x'_2}{y^3} - \mu_{\mathrm{c}}\frac{x'_2}{x^3} - \ddot{\theta}x'_1 + \dot{\theta}^2 x'_2 - 2\dot{\theta}\dot{x}'_1,\\
\ddot{x}'_3 &= -\mu_{\mathrm{s}}(1-\beta)\frac{x'_3}{y^3} - \mu_{\mathrm{c}}\frac{x'_3}{x^3},
\end{aligned}
\tag{1}
$$

where x'_n $(n = 1, 2, 3)$ are components of the position vector $\mathbf{x}$ of the particle in the CRS, $\mu_{\mathrm{s}} = Gm_{\mathrm{s}}$ is the Sun gravitational parameter, $\mu_{\mathrm{c}} = Gm_{\mathrm{c}}$ is the gravitational parameter of the comet; $r, \dot{\theta}, \ddot{\theta}$ are comet's heliocentric distance, the angular rate, and the angular acceleration about the Sun, respectively; $x = ({x'_1}^2 + {x'_2}^2 + {x'_3}^2)^{1/2}$ and $y = ((r + x'_1)^2 + {x'_2}^2 + {x'_3}^2)^{1/2}$.

AN ENERGY QUASI-INTEGRAL

Complex-analysis tools [2] permits one to derive an energy quasi-integral from Eqs. (1).

Denote by v the orbital velocity of the particle at the heliocentric distance y. Then the expression for the particle's energy quasi-integral assumes the form

$$
\frac{v^2}{2} = (1-\beta)\frac{\mu_{\mathrm{s}}}{y} + \frac{\mu_{\mathrm{c}}}{x} - \mu_{\mathrm{c}}\int_{t_0}^{t}(x'_1\dot{r} + \dot{\theta}x'_2 r)\frac{dt'}{x^3} + H_0,
\tag{2}
$$

where the constant

$$H_0 = \frac{v_0^2}{2} - (1-\beta)\frac{\mu_{\rm s}}{y_0} - \frac{\mu_{\rm c}}{x_0} \tag{3}$$

is the energy integral at the initial time t_0.

It is well known that

$$v^2 = |\mathbf{v}_{\rm ex} + \mathbf{v}'|^2 = v_{\rm ex}^2 + v'^{\,2} + 2\mathbf{v}_{\rm ex}\cdot\mathbf{v}', \tag{4}$$

where $\mathbf{v}_{\rm ex}$ denotes the reference frame velocity of a point fixed in the CRS and $\mathbf{v}'$ is the dust particle velocity relative to the CRS. So, if the velocity v' of the dust particle relative to the nucleus is equal to zero, then v^2 in the left side of the Eq. (2) can be replaced by $v_{\rm ex}^2$ and we obtain the so-called equation of the surfaces of zero relative velocity of the particle in the CRS:

$$\frac{2\mu_{\rm c}}{x} + 2(1-\beta)\frac{\mu_{\rm s}}{y} - 2\mu_{\rm c}\int_{t_0}^{t}(x_1'\dot{r} + \dot{\theta}x_2'r)\frac{dt'}{x(t')^3} - v_{\rm ex}^2 = C, \tag{5}$$

where $C = -2H_0$.

ESTIMATIONS AND DISCUSSION

It is useful to estimate terms in Eq. (2). For convenience, let us consider the minimal and maximal hypothetical comets. Take the nucleus radius of the minimal comet to be 0.4 km, as Sugano–Saiguse–Fujikawa Comet has, the radius of the maximal comet to be 20 km, as in the case of P/Schwassmann–Wachmann 1 Comet. Let us assume that the mean density of their nuclei equals $1\cdot10^3$ kg m^{-3}. Then masses of the nuclei are $2.681\cdot10^{11}$ kg and $3.351\cdot10^{16}$ kg, respectively. This leads to inequalities:

$$17.890\,\mathrm{m^3\,s^{-2}} \le \mu_{\rm c} \le 22.361\cdot10^5\,\mathrm{m^3\,s^{-2}}. \tag{6}$$

At the same time $\mu_{\rm s} = 13.273\cdot10^{19}$ m^3 s^{-2}.

In order to estimate the constant H_0, it is necessary to select a boundary in the coma out of which the dust particles are mainly decoupled from the outflowing gas – reach their terminal velocity. Early coma models set it at a few tens of the nucleus radius [3]. Also, temporary captured particles of submillimeter size and larger can be found in the circumnuclear volume of a mean radius of 100 the nucleus radii [4, 8]. In comparison, the data obtained in [10] indicate that within jets the boundary may be removed to a distance of a few thousand nucleus radius. Use for our purposes the distance $x_0 = 100$ the nucleus radii. Constant x_0 defines the distance to the starting position of the particle in Eq. (3). Let the starting particle have the position vector $\mathbf{y}_0$ along the Sun – comet axis. Then constant y_0 in Eq. (3) equals $y_0 = r_0 \pm x_0 = r_0 \pm |x_1'|_0$, where r_0 is the heliocentric distance of the cometary nucleus at the moment of the particle start. Because the value H_0 is being looked for at the single point $(r_0 + x_{1,0}',\, 0,\, 0)$, an acceptable approximation of $v_0^2/2$ can be obtained by putting

$$\frac{v_0^2}{2} = \frac{v_{\rm c,0}^2}{2} = \mu_{\rm s}\left(\frac{e-1}{2q} + \frac{1}{r_0}\right), \tag{7}$$

where $v_{\rm c,0}$ is the orbital velocity of the comet in an orbit with the eccentricity e and the perihelion distance q at the moment of the particle start. As to the value of β, for majority of materials $0 \le \beta < 1$. Only for iron, graphite or magnetite particles, and also for the fluffy ones, there is $\beta \ge 1$ [1, 11]. Using above numerical values shows that to first order it is sufficient to approximate H_0 by the expression:

$$H_0 = \mu_{\rm s}\left(\frac{e-1}{2q} + \frac{\beta}{r_0}\right). \tag{8}$$

It is seen that $H_0 < 0$ for $\beta < (1-e)r_0/2q$, that is only for comets in an elliptic orbit. Otherwise $H_0 \ge 0$. On the whole, when $\beta < (1+e)/2$, there is a region of r_0 on an elliptic orbit where $H_0 \le 0$.

To estimate the integral in Eq. (2), assume that a particle is started from a surface $\mathbf{x} = \mathbf{x}_0$ around a comet at $t = t_0$ and arrived at a point $(x_{1*}', x_{2*}', x_{3*}')$ at $t = t_*$ with the relative speed $v' = 0$. Let at this point a net zero acceleration act on the particle in the rotating frame CRS. Thus, the particle will remain in this equilibrium point of the CRS at least by moment $t > t_*$. Then the integral from Eq. (2) may be broken up into the sum:

$$I \equiv \int_{t_0}^{t}(\dot{r}x_1' + r\dot{\theta}x_2')\frac{dt'}{x^3} = \int_{t_0}^{t_*}(\dot{r}x_1' + r\dot{\theta}x_2')\frac{dt'}{x(t')^3} + \int_{t_*}^{t}(\dot{r}x_{1*}' + r\dot{\theta}x_{2*}')\frac{dt'}{x_*^3}. \tag{9}$$

Table 1. Results of Estimations of Terms in Equation (2)

Comet:	46P/Wirtanen	C/1995 O1(Hale–Bopp)	29P/Schwassmann–Wachmann 1
$v_{\rm ex}^2/2$	4.211	5.614	0.761
$\mu_{\rm s}/y$	5.614	5.617	1.490
$\mu_{\rm c}/x$	$2.301{\cdot}10^{-15}$	$3.952{\cdot}10^{-12}$	$6.139{\cdot}10^{-12}$
$-\mu_{\rm c}I$	$1.899{\cdot}10^{-9}$	$2.475{\cdot}10^{-7}$	$2.732{\cdot}10^{-6}$
$H_0(\beta)$	$6.719\beta{-}1.389$	$7.184\beta{+}0.046$	$1.517\beta{-}0.743$

Note. All the quantities are expressed in terms of $10^8\,{\rm m^2\,s^{-2}}$ unit. The quantities in the right-hand side of Eq. (2) are calculated at the point $(r(\theta=80^o)+x'_{10}, x'_{10}, x'_{10})$, where $x'_{10}\equiv(x'_1)_0=-100$ nuclear radii. The fourth order Runge–Kutta method was used to solve the system of the differential equations (1) at the calculation of the integral I in Eq. (9). The value of H_0 is estimated at the point $(r(\theta_0=60^o)+x'_{10}, x'_{10}, x'_{10})$, with the relative starting velocity $v'_0=(\dot x'_{10}, \dot x'_{20}, \dot x'_{30})=(140,-14,14)\,{\rm m\,s^{-1}}$. The orbit parameters e, q are taken from [http: //cfa-www.harvard.edu/iau/Ephemerides/Comets/]. The epoch of osculation of the orbital elements is 2003 June 10.0 TT. It is supposed that the mean density of the comet nuclei is equal to $1\cdot10^3\,{\rm kg\,m^{-3}}$, the mass of Comet Hale–Bopp is $2.1\cdot10^{15}$ kg, the mean radius of Comet 46P/Wirtanen is 690 m, and that of Comet Schwassmann–Wachmann 1 is $2\cdot10^4$ m.

To evaluate the first integral in the right-hand side of this relation, differential Eqs. (1) must be solved. But the second integral may be written in the form

$$I_*\equiv\frac{x'_{1*}}{x_*^3}\int_{t_*}^t\dot r dt'+\frac{x'_{2*}}{x_*^3}\int_{t_*}^t r\dot\theta dt'=\frac{x'_{1*}}{x_*^3}[r(t)-r(t_*)]+p\frac{x'_{2*}}{x_*^3}\int_{\theta_*}^{\theta}\frac{d\vartheta}{1+e\cos\vartheta}\,, \tag{10}$$

where p is the semilatus rectum of the comet's orbit, and $\theta_*=\theta(t_*)$ is the true anomaly of the comet at $t=t_*$. Performing the quadrature yields the formulas

$$\int_{\theta_*}^{\theta}\frac{d\vartheta}{1+e\cos\vartheta}=\begin{cases}2e_*^{-1}(\pi_*+\arctan A), & (e<1,\ e_*=\sqrt{|1-e^2|}),\\ e_*^{-1}\ln[(1+A)/(1-A)], & (e>1,\ e_*=\sqrt{|1-e^2|}),\\ 2A, & (e=1,\ e_*=1),\end{cases} \tag{10$_*$}$$

where

$$\pi_*=\begin{cases}0, & \text{if}\quad \tan(\theta/2)\tan(\theta_*/2)>(e+1)/(e-1),\\ \mathrm{sign}(\tan(\theta/2))\cdot\pi, & \text{if}\quad \tan(\theta/2)\tan(\theta_*/2)<(e+1)/(e-1),\end{cases}$$

and

$$A=\frac{e_*\cdot\sin((\theta-\theta_*)/2)}{\cos((\theta-\theta_*)/2)+e\cdot\cos((\theta+\theta_*)/2)}.$$

Table 1 lists illustrative results of the estimations for some of the representative comets.

STANDARD TREATMENT OF ENERGY QUASI-INTEGRAL

Another interpretation of Eq. (2) arises after introducing some habitual quantities.

Consider the energy quasi-integral (2). Since $|\mathbf{v}|\equiv v\geq0$, the left member of Eq. (2) is greater than or equals zero. Therefore, for the right-hand side of the equation, the following condition is satisfied:

$$(1-\beta)\frac{\mu_{\rm s}}{y}+\frac{\mu_{\rm c}}{x}-\mu_{\rm c}\int_{t_0}^t(x'_1\dot r+\dot\theta x'_2 r)\frac{dt'}{x^3}+H_0\geq0. \tag{11}$$

Define

$$E_0=m_pH_0,\quad T_0=m_pv_0^2/2, \tag{12}$$

$$U=-m_p\left[(1-\beta)\frac{\mu_{\rm s}}{y}+\frac{\mu_{\rm c}}{x}-\mu_{\rm c}\int_{t_0}^t(x'_1\dot r+\dot\theta x'_2 r)\frac{dt'}{x^3}\right], \tag{13}$$

where m_p is the mass of the dust particle. Then the condition (11) becomes

$$E_0 - U \geq 0, \quad E_0 = T_0 + U_0, \tag{14}$$

where $U_0 = U|_{t=t_0}$. According to the standard treatments of the introduced quantities, mechanical motion of the particle is possible in a space region where $U \leq E_0$, that is where inequality (14) is fulfilled. If at once the initial energy of the particle $E_0 < 0$, the motion is limited to the condition $U = E_0$. This equality defines the surfaces of zero relative velocity v' of the dust particle motion by means of Eq. (5). If under condition (14) $E_0 > 0$ (that is $H_0 > 0$) then the motion of the particle is unrestricted in space.

As can be seen from Table 1, any value of β leads to $H_0 > 0$ for Comet Hale–Bopp when a particle is started under the conditions listed in the *Note*.

Acknowledgements. Dr. Yu. V. Babenko is gratefully acknowledged for eagerness to help in a numerical integration of the system of the differential equations of a dust particle motion. Unfortunately, author picked out another calculus software.

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