THE CONSTRAINTS ON THE POWER SPECTRUM OF RELIC GRAVITATIONAL WAVES FROM CURRENT OBSERVATIONS OF LARGE-SCALE STRUCTURE OF THE UNIVERSE

B. Novosyadlyj, S. Apunevych

Astronomical Observatory, Ivan Franko National University of Lviv
8 Kyryla i Methodia Str., 79005 Lviv, Ukraine
e-mail: novos@astro.franko.lviv.ua, apus@astro.franko.lviv.ua

Within the framework of cosmological inflationary models, relic gravitational waves arise as a natural consequence of the quantum nature of primordial space-time metric fluctuations and a validity of the general theory of relativity. Therefore, the detection of cosmological gravitational waves would be the strongest evidence in support of basic assumptions of the inflation theory. Although the tracing gravitational waves by polarization patterns on last scattering surface of the cosmic microwave background is only planned for forthcoming experiments, some general constraints on the tensor mode of metric perturbations (*i.e.*, gravitational waves) can be established right now. We present a determination of the amplitude of the relic gravitational waves power spectrum. An indirect best-fit technique was applied to compare observational data and theoretical predictions. As observations we have used data on the large-scale structure of the Universe and anisotropy of the cosmic microwave background temperature. The conventional inflationary model with 11 parameters has been investigated, all parameters were jointly evaluated. This approach gave us a possibility to find parameters of the power spectrum of gravitational waves along with statistical errors.

INTRODUCTION

For the last decade, due to the progress in observations, the cosmology entered a new stage of its development. This stage was signalized by a new generation of experiments aimed to measurements of anisotropies of the cosmic microwave background (CMB). These were the balloon-borne BOOMERanG, MAXIMA, Archeops, the ground-based interferometers DASI, CBI, VSA. And probably, the most important one is the successor of the COBE space mission, Wilkinson Microwave Anisotropy Probe (WMAP) that published in 2003 the one-year observation results. WMAP has carried out measurements of the CMB over the whole sky with an unprecedented angular resolution and high sensitivity of detectors. The data and Web-links for these experiments are available at the Web-site of the Legacy Archive for the Microwave Background Data Analysis (LAMBDA) project [5]. The CMB explorations were complemented with extensive studies of expansion dynamics of the Universe by means of measuring distances to Supernovae and large-scale structure surveys.

Advances in the quality of experimental data manifestly call for a model capable to explain a whole set of collected data. Now, it is well understood that the simplest cosmological models cannot match the observations adequately, as, *e.g.*, the standard flat CDM model with a scale-invariant power spectrum of density fluctuations. The observations advance the complication of the model, the number of parameters increases as well as the number of phenomena encompassed by the theory. Today, the elaborate inflationary models need about 11 parameters for the proper description of the reality.

Cosmological parameters could be classified as follows:

- The parameters related to a background model of a homogeneous and isotropic Universe. The evolution of the Universe in this model is determined by the amount of energy densities of different components in ratio to the critical one $\Omega_i = \rho_i/\rho_{cr}$. These are densities of the baryon component Ω_b, hot dark matter (massive neutrinos) Ω_ν, cold dark matter Ω_{cdm}, and the density parameter for dark energy (*i.e.*, the cosmological constant or quintessence) Ω_{de}.
- The global properties of the Universe. According to the Friedmann equations, sum of density parameters gives a unity for a spatially flat Universe, or $1 - \Omega_k$ for the curved one, so Ω_k is a curvature parameter. The Hubble constant $\mathrm{H}_0 = 100\,h$ km s^{-1} Mpc^{-1} is regarded as a global parameter too.

- Parameters associated with the inflation. According to an inflationary scenario, the large-scale structure of the Universe is assumed to be formed due to the growth of primordial matter density perturbations because of the gravitational instability. The perturbations have an adiabatic nature and originate from quantum fluctuations stretched to the cosmological scales during the stage of the exponential expansion – the inflation. Perturbations are usually described by their power spectra in the most general form $P_s(k) = A_s k^{n_s}$ for scalar and $P_t(k) = A_t k^{n_t}$ for tensor mode of perturbations.
- The list of parameters has some additional parameters, like the late re-ionization parameter τ_c and biasing parameters to relate the distribution of density to the spatial distribution of real astrophysical objects, *e.g.*, galaxies.

This parameter set is required to get the model predictions which are to be compared with the observations in order to find out how good the theoretical assumptions match the results of current observations. As far as these predictions depend on the parameters quite nonlinearly, the determination of the best-fit value for one parameter requests the determination of a full set of parameters, by means of statistics. Actually that is the main task of this investigation, sometimes referred as a testing of cosmological models.

In this paper, we shall pay attention to one parameter among many others, namely, the primordial power spectrum of tensor mode of space-time metric perturbations (relic gravitational waves). We shall use the most common kind of cosmological model and analyse some particular inflation models. Sometimes, this model is designated as a concordance model, or even standard model. The purpose of this model is to give a plausible explanation for the large-scale structure of the Universe and its global properties.

THE NATURE OF RELIC GWs

Indeed, gravitational waves are very simple conceptually but proved to be extremely elusive for the detection. Gravitational waves ought to be emitted by any physical system with changes in the quadrupole distribution of the stress-energy density. This follows from equations of general relativity as a free solution (wave solution) for small linear space-time metric perturbations on the background metric.

Obviously, only astrophysical objects could produce gravitational waves of significant amplitudes. The number of probable astrophysical sources is under discussion for observational programmes (see for review [4]). There are by now no direct successful detections of any kind of sources. For cosmological GWs, in fact, a very small chance exists to be detected by any human-made antenna because of their super-long wavelengths.

The cosmological GWs play the role of the most ancient relic in our Universe. They have their origin in the first moments of the Universe evolution, presumably the times of inflation. If we assume the general theory of relativity to be valid in those times and quantum zero-point oscillations to exist, so we come to conclusion about the inescapable existence of GWs. A variable gravitational field in a very early Universe parametrically amplifies quantum oscillations, making up a stochastic gravitational background. The amplitude of these GWs is determined by energy scales at the moment of emission, so for inflation this amplitude determines the energy scale of processes that give a rise for inflation. So, the nature of relic gravitational waves is very fundamental for physics. In fact, both scalar mode and tensor modes of space-time metric perturbations share a common origin from a single physical process – quantum oscillations.

The subsequent evolution of metric linear perturbations is completely described by the gauge-invariant formalism for cosmological perturbations. The relic GWs are represented by tensor mode of space-time metric perturbations on the isotropic and homogeneous expanding background. As far as GWs do not interact with the rest of the medium in an expanding Universe, their amplitude should decrease in course of time (see [1] for an exhaustive review).

The gauge-invariant theory leads to the equation for the evolution of tensor perturbations of space-time metrics in vacuum or a perfect fluid:

$$\ddot{\mathrm{H}}^{(T)} + 2\frac{\dot{a}}{a}\dot{\mathrm{H}}^{(T)} + (2\mathcal{K} + k^2)\mathrm{H}^{(T)} = 0, \tag{1}$$

where $\mathrm{H}^{(T)}$ is the gauge-invariant amplitude of tensor perturbations for two polarizations, a is a scale factor, $\mathcal{K} = (-1, 0, 1)$ is the curvature index, and k is the wavenumber. The dots denote the derivatives with respect to the conformal time η. The solution of this equation is a propagating damped wave.

OUTLINE OF THE METHOD

As it was stated above, both tensor and scalar modes of space-time metric perturbations have the same origin. The perturbations of scalar mode are connected to the density perturbations so they eventually lead to the formation of galaxies, clusters, and voids, *i.e.*, the large-scale structure of the Universe. Since tensor perturbations do not lead to the formation of the large-scale structure, cosmological GWs can not be revealed by the present state of the observable structure. Both scalar and tensor perturbations produce the power in the CMB angular power spectrum due to the Sachs–Wolfe effect. Of course, because of its specific polarization properties, the relic GWs should generate a particular polarization pattern of CMB anisotropies, but the detection of polarization patterns in CMB is the matter of the future.

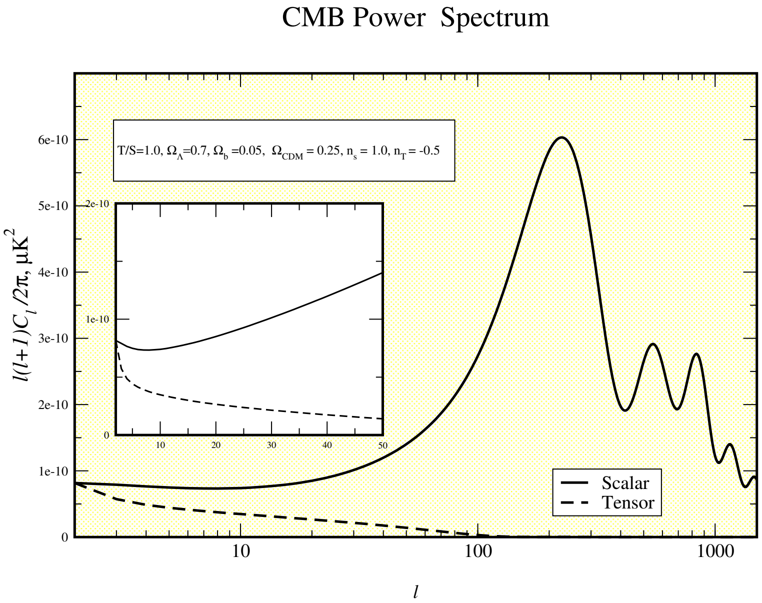

Figure 1. The scalar and tensor contributions to the CMB angular power spectrum for a particular model

Therefore, we propose an indirect method. As far as the CMB power spectrum consists of contributions from scalar and tensor modes, we can extract the last one if we precisely know the scalar contribution from the large-scale structure. Also, the relic GWs have extra-long wavelengths of a particle horizon size, so the CMB power spectrum from them should manifest a fast decrease with higher multipole moments (smaller scales). On the contrary, the density perturbations at larger multipoles produce a feature-rich CMB power spectrum, a series of acoustic peaks. Thus, the physical properties of tensor mode of cosmological perturbations of space-time metric can be estimated on the basis of observational data on an angular power spectrum of CMB combined and the large-scale structure of the Universe in a wide range of scales – from the galactic ones to the size of a particle horizon, from 10^{-3} to 10^4 Mpc. This way, we can constrain a total level of the intensity of gravitational waves. In the multipoles range of $\ell \sim 10-20$, the scalar and tensor contributions are comparable (Fig. 1). So, the precision of the results is to be determined by the precision of datapoints within this range. This method is basically a normalization of the angular power spectrum.

THE BUILDING PREDICTIONS

Unfortunately, the cosmological models are quite complicated and the most of predictions could not be reduced to analytical functions with cosmological parameters as arguments. Therefore, for the given parameter set, one has to run numerical computations to make predictions. One way is the "brute-force" method, when the numerical packages like CMBfast [6] are utilized. Since the testing requires a large number of models to be evaluated, so even quite fast codes need a huge computational resources to accomplish the task. Another demerit lies in the so-called "black box" uncertainty of numerical calculations, when the physical processes are hidden inside computations. So, we propose semi-analytical approaches to make predictions of the CMB power spectrum.

To obtain a tensor contribution to the CMB power spectrum within a low multipoles range ($2 < \ell < 20$), we have developed an analytical approximation able to reproduce a computed spectrum with a high accuracy, for a wide range of parameters values. This approximation has the form:

$$\ell(\ell+1)\,\mathcal{C}_\ell^T = \frac{A(n_t,\Omega_0)}{l+b(n_t,\Omega_0)} \cdot K_\ell(\Omega_{de}) \times \exp(-C(n_t,\Omega_0)\cdot\ell^2 + D(n_t,\Omega_0)\cdot\ell), \qquad (2)$$

where the coefficients A, b, C, and D represent polynomial functions of n_t and $\Omega_0 = 1 - \Omega_k$, $K_\ell(\Omega_{de})$ is the amplification factor for the CMB power spectrum caused by dark energy. Analytical formulae for them are presented in our paper [7].

A semi-analytical approximation for the scalar contribution to the CMB power spectrum for the multipoles range $2 < \ell < 20$ was developed in our previous paper [3] and is used here. This approximation combines the speed of computations and a clear unobscured physical meaning of processes that cause the anisotropies in CMB. The angular power spectrum of CMB at higher multipoles has a number of peaks separated with dips and manifests the damping of an average amplitude to the highest ℓ. This features are explained by acoustic oscillations in a photon–baryon fluid set up by adiabatic perturbations at the entering the sound horizon. Instead of calculations of power for each ℓ within a wide range, we propose to describe the whole shape of the CMB power spectrum by positions and amplitudes of the peaks and dips. The insignificant loss of information will be a price for the computations speed-up. The number of analytical approximations was developed for heights and positions of first three peaks and for position of the first dip in the CMB power spectrum in the models considered, see [3].

OBSERVATIONAL DATA

Of course, the results of testing of the model strongly depend on the data used. The next important requirement the data should meet is the statistical independence, *i.e.*, the covariance matrix of datapoints errors should be diagonal. Here we have used the same set of observational data as in [2, 3], with some changes. The amplitudes and positions of the 1st and 2nd acoustic peaks and position of the first dip have been taken from the results of the WMAP mission team [8], position and height of the third peak have been taken from the last recompilation of the results of the BOOMERanG experiment [10]. At large angular scales (a lower multipoles range), we use all datapoints from the COBE experiment and WMAP [5], except dipole.

The CMB data are complemented by the large-scale structure data: mass function and spatial distribution of rich galaxy clusters, temperature function of X-ray clusters, Ly-α forests of absorption lines in distant quasars spectra, peculiar velocities of galaxies. These LSS data can establish the amplitude and shape of a spatial power spectrum of matter density perturbations. During the determination of parameters we have the problem with the degeneracy in dependence of observational manifestations upon the parameters. This problem is partially eliminated when we add additional measurements to the observational set: determinations of expansion dynamics due to angular distances to Supernovae of Ia type, independent determinations of the Hubble constant, constraints on the baryon content from the Big Bang nucleosynthesis theory.

There are in all 41 values in the list of observational datapoints along with 1σ statistical errors of their measurements. We consider all measurements to be independent and take their probability distribution as a normal distribution. These values and errors are described in details in the cited papers. Thus, if we compare these results with those from [3], we obtain the answer to the question how improvements in measurements of CMB anisotropies affect the determination of parameters of the observable Universe.

LIKELIHOOD ANALYSIS

Let us have N observational characteristics and we are searching the best-fit values of n cosmological parameters. In other words, we have the parameter set:

$$\overrightarrow{P} = (\Omega_{CDM}, \Omega_{de}, \Omega_\nu, N_\nu, \Omega_b, h, A_s, n_s, A_t, n_t, \tau_c)$$

and the set of observations:

$$\overrightarrow{D} = (A_{p_1}, \ell_{p_1}, A_{p_2}, \ell_{p_2}, A_{p_3}, \ell_{p_3}, \ell_{d_1}, C_{l_{low}}, LSS, NS, h, SN).$$

To find the best-fit values for 11 parameters we are using the Levenberg–Marquardt algorithm [9] to minimize the function

$$\chi^2 = \sum_{j=1}^{N} \left(\frac{\tilde{y}_j - y_j}{\Delta \tilde{y}_j} \right)^2, \tag{3}$$

where $\tilde{y}_j$ is the observational value for some j-th characteristic, y_j is its theoretically predicted value, $\Delta\tilde{y}_j$ is a statistical uncertainty for the measured value. Since the number of neutrino species N_ν is a discrete value, so we were searching for values of 10 parameters at N_ν held fixed at 1, 2, 3. As in the previous papers, instead of dimensional values of the amplitude of scalar mode power spectra A_s, we have used the dimensionless value δ_h, defined as rms of fluctuation of the matter density on scale of the present horizon of particle, $A_s = 2\pi^2\delta_h^2(3000\,\mathrm{Mpc}/h)^{3+n_s}$.

To find errors for parameters values one has to carry out an exploration of the likelihood function profile $\mathcal{L} \propto e^{-\frac{1}{2}\chi^2}$ in a parametric space to determine confidential ranges (marginalizing). The estimations of confidential ranges for parameters are mostly based on the direct likelihood function integration [9], a kind of cumbersome computations. In [2, 3] we have proposed "economical" methods for the marginalization procedure to avoid the direct integration. Here we propose a combined approach: the likelihood function profile for parameter x_k is built using the minimization of χ^2 in a subspace of $n-1$ parameters

$$\mathcal{L}(x_k) = e^{-\frac{1}{2}[\chi^2(x_{i\neq k}^{bf}, x_k) - \chi^2_{min}]}\,, \tag{4}$$

where $x_{i\neq k}^{bf}$ are the best-fit values of cosmological parameters ($i = 1, 2, \ldots, 12,\ i \neq k$), for which χ^2-function has the minimum at a fixed value of parameter x_k. This kind of representation for $\mathcal{L}(x_k)$ can be obtained from the general integral form. A numerical experiment proves that the results from the proposed function $\mathcal{L}(x_k)$ for a confidential level estimation virtually coincide with the values obtained by integration.

RESULTS

For historical reasons, the common practice is to give a description of relic gravitational waves by the ratio of amplitudes of tensor T and scalar contributions S to the quadrupole ($\ell = 2$) component of the CMB power spectrum. However, probability distributions for A_t and T/S differ, and $\mathcal{L}(A_t)$ function is closer to the Gaussian shape than $\mathcal{L}(T/S)$. So, we are using $\mathcal{L}(A_t)$ to determine upper bounds and to recalculate it then to T/S. We define the upper bound $A_t^{2\sigma}$ at a confidential level 2σ as a value for which the square under the curve $\mathcal{L}(A_t)$ has 95.4% of the total square under this curve from 0 to ∞.

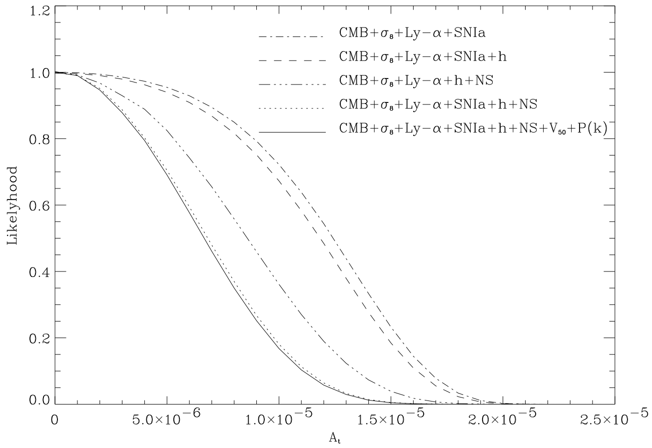

Figure 2. Likelihood functions $\mathcal{L}(A_t) = \exp[-\frac{1}{2}\chi(A_t)]$ for various sets of observational data

In terms of our technique, CMB and LSS data have an insufficient statistical weight in order to provide a simultaneous determination of both amplitude and slope of the tensor power spectrum. In other words, if we would leave both A_t and n_t as free parameters, then A_t could take an arbitrary large value under condition $n_t \to -\infty$. If we keep the lower bound for n_t fixed, then the upper bound for A_t depends on it. This problem finds its natural explanation by limitations of the indirect method used. We have no data to distinguish the amplitude and slope separately. Fortunately, the majority of inflation models relate the n_t with the slope of the scalar power spectrum n_s. So, we have also analysed the likelihood functions for the same observational data and some generic inflation models: with a flat spectrum of tensor mode ($n_t = 0$), natural inflation with $n_t = n_s - 1$, and chaotic inflation with $n_t = 0.5(n_s - 1)$. The upper 2σ constraints for them are the following: $A_t^{2\sigma} = 1.9 \cdot 10^{-5}$ for the first model and $A_t^{2\sigma} = 1.0 \cdot 10^{-5}$ for the rest. The corresponding values for them are $T/S = 0.6$ and 0.18. These three models are more interesting from the standpoint of manifestations of tensor mode in the data of observations, so further analysis will include merely them. As far as the likelihood functions $\mathcal{L}(A_t)$ for them are very similar, it is quite enough to analyse one of them, namely, we take a model with $n_t = 0.5(n_s - 1)$.

Figure 2 illustrates how observational data influence on the half-width of a likelihood function. As we can see, the adding the observational data with the constraints on cosmological parameters themselves, like the Hubble constant, Big Bang Nucleosynthesis, SNIa observations, and data on the large-scale structure, decreases the confidence level for the models with high tensor amplitudes. At the confidential level of 2σ (95.4%), this amplitude cannot exceed $\sim 20\%$ of the scalar mode amplitude. In our previous estimations (see [3]), based on the data of the balloon experiment BOOMERanG, this limitation was almost four times larger (at 1σ C.L. the ratio was estimated as $T/S = 1.7$ for the model with free n_t). So, we clearly see the achievements of WMAP with high precision and covering over all sky.

The results for a whole parameter set are summarized in Table 1. The constraints are tabulated here for the case of a chaotic inflation. As we can see, the accordance of these best-fit values with previous determinations [2, 3] is quite satisfactory. The new precise measurements of CMB temperature anisotropies significantly tightened confidential ranges and lowered upper constraints for Ω_ν, τ_c and T/S. The results obtained agree with determinations of other authors which used the WMAP data [11] (different definitions).

Table 1. Best-fit values of parameters, lower and upper bounds at 2σ (95.4%) confidential level

	Ω_{de}	Ω_m	Ω_ν	Ω_b	h	δ_h	n_s	T/S	τ_c
Best-fit	0.61	0.41	0	0.062	0.61	$4.2 \cdot 10^{-5}$	0.92	0	0
Lower bound	0.52	0.31	0	0.046	0.52	$3.6 \cdot 10^{-5}$	0.89	0	0
Upper bound	0.69	0.51	0.03	0.078	0.71	$5.2 \cdot 10^{-5}$	0.98	0.6	0.15

CONCLUSIONS

From the standpoint of statistics an interpretation of available observational data on large-scale structure of the Universe and CMB does not require the presence of a considerable amount of relic gravitational waves. According to the basic assumptions of the early Universe physics, these relic waves have an inescapable nature and common origin with matter density fluctuations seeding a large-scale structure. Within the framework of the inflation theory, the manifestations of the gravitational background can be quite prominent, as long as the amplitude of relic GWs directly connected to energy scales of inflation, *e.g.*, the Grand Unification Theory energy scale. Thus, the upper bounds on the amplitude of GWs appear to be utmost important to provide constraints on the moment and energy scales of the inflation allowing to discriminate among models of inflation.

The advances in measurements of the CMB in the space experiment WMAP substantially lowered the upper bound for the amplitude of tensor mode of perturbations (*i.e.*, relic gravitational waves) to the level of $T/S \leq 0.6$ (95.4% C.L.) for models with the free slope parameter n_t. For models with a flat power spectrum of gravitational waves ($n_t = 0$), or some close to that spectra ($n_t \sim 1 - n_s$), this limit appears to be even lower, $T/S \leq 0.18$ (95.4%).

[1] *Durrer R.* The theory of CMB anisotropies // J. Phys. Studies.–2001.–**5**, N 2.–P. 177–215.

[2] *Durrer R., Novosyadlyj B.* Cosmological parameters from complementary observations of the Universe // Mon. Notic. Roy. Astron. Soc.–2001.-**324**.–P. 560–572.

[3] *Durrer R., Novosyadlyj B., Apunevych S.* Acoustic Peaks and Dips in the Cosmic Microwave Background Power Spectrum: Observational Data and Cosmological Constraints // Astrophys. J.–2003.–**583**, N 1.–P. 33–48.

[4] *Grishchuk L. P. et al.* Gravitational wave astronomy: in anticipation of first sources to be detected // Usp. Fiz. Nauk.–2001.–**171**.–P. 3–59.

[5] [`http://lambda.gsfc.nasa.gov/`].

[6] [`http://www.cmbfast.org`].

[7] *Novosyadlyj B., Apunevych S.* Cosmological constraints on the amplitude of relic gravitational waves // J. Phys. Studies.–2005.

[8] *Page L. et al.* First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Interpretation of the TT and TE Angular Power Spectrum Peaks // Astrophys. J. Suppl. Ser.–2003.–**148**.–P. 233–241.

[9] *Press, W. H. et al.* Numerical recipes in FORTRAN.–New York: Cambridge Univ. Press., 1992.

[10] *Ruhl J. E. et al.* Improved Measurement of the Angular Power Spectrum of Temperature Anisotropy in the Cosmic Microwave Background from Two New Analyses of BOOMERANG Observations // Astrophys. J.–2003.–**599**.–P. 786–805.

[11] *Spergel D. N. et al.* First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters // Astrophys. J. Suppl. Ser.–2003.–**148**.–P. 175–194.