STABILITY OF BALLOONING MODES IN THE INNER MAGNETOSPHERE

O. S. Burdo[1], **O. K. Cheremnykh**[1], **O. P. Verkhoglyadova**[2]

[1] *Space Research Institute NASU & NSAU*
40 Acad. Glushkova Avenue, 03022 Kyiv, Ukraine

[2] *Taras Shevchenko National University*
6 Acad. Glushkova Avenue, 03022 Kyiv, Ukraine

In the present work the problem of ideal MHD pressure driven ballooning modes stability with finite compressibility plasma of the magnetosphere is studied. The set of equations described these modes is derived for the dipole model of magnetic field of the Earth. It is shown that the stability boundary of these modes is essentially affected by value and derivative of pressure. Numerical results are presented.

BASIC EQUATIONS

As starting point for formulation of problem we consider the following set of small oscillation equations

$$\rho \frac{d\vec{v}}{dt} = -\nabla\tilde{p} + \frac{1}{c}\left(\vec{j}\times\tilde{\vec{B}}\right), \quad \tilde{\vec{B}} = \mathrm{rot}\left(\vec{\xi}\times\vec{B}\right), \quad \tilde{p} = -\left(\vec{\xi}\nabla p\right) - \gamma p\,\mathrm{div}\,\vec{\xi}, \tag{1}$$

which describe arbitrary MHD perturbations of ideal compressible plasma. Here we use the generally accepted notations for all physical values. Tilde indicates the perturbed component of a corresponding value.

By taking into account the relations

$$\vec{j} = (c/4\pi)\,\mathrm{rot}\,\vec{B}, \quad \vec{v} = \partial\vec{\xi}\Big/\partial t \tag{2}$$

the system of Eqs. (1) can be reduced to the following vector equation for the displacement vector $\vec{\xi}$

$$\rho\frac{\partial^2\vec{\xi}}{\partial t^2} = \nabla\left(\vec{\xi}\cdot\nabla\right)p + \gamma\nabla\left(p\,\mathrm{div}\,\vec{\xi}\right) + \frac{1}{4\pi}\left(\mathrm{rot}\vec{B}\times\tilde{\vec{B}}\right). \tag{3}$$

The first and second right terms in this equation are important for existence of pressure driven modes.

With help of the relation

$$\vec{T} \equiv \tilde{\vec{B}} + \frac{\vec{j}\times\nabla\psi}{|\nabla\psi|^2}\left(\vec{\xi}\nabla\psi\right) = T_1\nabla\psi + T_2\frac{\vec{B}\times\nabla\psi}{|\nabla\psi|^2} - T_3\vec{B}, \tag{4}$$

$$\vec{\xi} = \xi\frac{\nabla\psi}{|\nabla\psi|^2} + \eta\frac{\vec{B}\times\nabla\psi}{\left|\vec{B}\right|^2} + \tau\frac{\vec{B}}{\left|\vec{B}\right|^2},$$

where ψ marks magnetic surfaces (see [1]), we rewrite Eq. (3) in the following useful forms

$$\frac{1}{\left|\vec{B}\right|^2|\nabla\psi|^2}\rho\frac{\partial^2\xi}{\partial t^2} = \frac{\nabla\psi\cdot\nabla T_3}{|\nabla\psi|^2} + \frac{1}{\left|\vec{B}\right|^2}\left[\frac{\nabla\psi\cdot\nabla T_0}{|\nabla\psi|^2} - 2p'T_3 + T_2\left(S-\gamma_s\right) + \vec{B}\cdot\nabla T_1 + K\xi\right], \tag{5}$$

$$\frac{|\nabla\psi|^2}{\left|\vec{B}\right|^2}\rho\frac{\partial^2\eta}{\partial t^2} = \frac{\vec{B}\times\nabla\psi}{\left|\vec{B}\right|^2}\nabla T_0 + \vec{B}\cdot\nabla T_2 + \left(\vec{B}\times\nabla\psi\right)\nabla T_3, \quad \rho\frac{\partial^2\tau}{\partial t^2} = \vec{B}\cdot\nabla T_0,$$

where the expressions T_i, S, α_s, γ_s, K and $(...)'$ are given by

$$T_1 \;=\; \frac{\vec{B}\cdot\nabla\xi}{|\nabla\psi|^2}, \quad T_2 \;=\; \frac{1}{\alpha_s}\left[\vec{B}\cdot\nabla\eta \,+\, (\gamma_s - S)\,\xi\right],$$

$$T_3 \;=\; \frac{\nabla\psi\cdot\nabla\xi}{|\nabla\psi|^2} \;+\; \frac{\vec{B}\times\nabla\psi}{\left|\vec{B}\right|^2}\nabla\eta \;+\; \frac{\xi}{\left|\vec{B}\right|^2}\left[\mathrm{div}\;(\alpha_s\,\nabla\psi) \,+\, 2\,p'\right], \quad S \;=\; \frac{\vec{B}\times\nabla\psi}{|\nabla\psi|^2}\,\mathrm{rot}\,\frac{\vec{B}\times\nabla\psi}{|\nabla\psi|^2}, \tag{6}$$

$$\alpha_s \;=\; \frac{\left|\vec{B}\right|^2}{|\nabla\psi|^2}, \quad \gamma_s \;=\; \frac{\vec{j}\cdot\vec{B}}{|\nabla\psi|^2}, \quad K \;=\; \frac{\alpha_s}{\gamma_s}\,(\gamma_s - S) \;+\; \frac{p'}{\left|\vec{B}\right|^2}\,\frac{\nabla\psi\cdot\nabla\left(2p + \left|\vec{B}\right|^2\right)}{|\nabla\psi|^2}, \quad (...)' \;\equiv\; d\,(...)/d\,\psi.$$

Eqs. (5), (6) describe the general structure of MHD perturbations and must satisfy equilibrium plasma configuration conditions.

MODEL

We consider magnetospheric plasma equilibrium

$$\nabla p \;=\; \vec{j}\times\vec{B}\Big/c \tag{7}$$

in right-handed spherical coordinate system $(r,\;\vartheta,\;\varphi)$ and use dipole model of the geomagnetic field, which can be presented in these coordinates in the form

$$\vec{B} \;=\; \nabla\varphi\times\nabla\Psi, \quad \Psi \;=\; M\,\cos^2\vartheta/r, \tag{8}$$

where M is the Earth magnetic dipole moment.

We also omit for simplicity the longitudinal current

$$\vec{j}\cdot\vec{B} \;=\; 0. \tag{9}$$

Then from condition of current closing we obtain the relation

$$\left(\vec{B}\times\nabla\psi\right)\nabla\left|\vec{B}\right| \;=\; 0. \tag{10}$$

REDUCTION OF THE EQUATIONS

To carry out the numerical calculation we reduce the Eqs. (5) as follows. At first we put in Eqs. (5)

$$\psi \;=\; \Psi, \tag{11}$$

and from Eqs. (9) and (10) we obtain

$$S \;=\; \gamma_s \;=\; 0. \tag{12}$$

Then we rewrite Eqs. (5) for ballooning perturbations

$$\frac{|\nabla\psi\cdot\nabla X|}{|\nabla\psi|^2}, \quad \frac{\left|\left(\vec{B}\times\nabla\psi\right)\cdot\nabla X\right|}{\left|\vec{B}\right|\,|\nabla\psi|} >> \frac{X}{b}, \quad \frac{\left|\vec{B}\cdot\nabla X\right|}{\left|\vec{B}\right|}, \tag{13}$$

where b is the spatial scale of problem ($b \sim R_E$), R_E is the Earth radius, X is perturbed value.

The accurate calculations lead us to equations

$$\frac{\omega^2\rho}{|\nabla\psi|^2}\,\xi \;+\left(\vec{B}\cdot\nabla\right)\left[\frac{1}{|\nabla\psi|^2}\,\left(\vec{B}\cdot\nabla\right)\,\xi\right] \;-\; \frac{2\,r}{M\,a^2}\,(T_0 + p'\xi) \;=\; 0, \tag{14}$$

$$\frac{\omega^2 \left|\nabla\psi\right|^2 \rho}{\left|\vec{B}\right|^2}\,\eta + \left(\vec{B}\cdot\nabla\right)\left[\frac{\left|\nabla\psi\right|^2}{\left|\vec{B}\right|^2}\left(\vec{B}\cdot\nabla\right)\eta\right] = 0, \quad \rho\,\omega^2\tau + \vec{B}\cdot\nabla T_0 = 0,$$

where

$$T_0 = \frac{\gamma p \left|\vec{B}\right|^2}{\gamma p + \left|\vec{B}\right|^2}\left(\operatorname{div}\frac{\tau\,\vec{B}}{\left|\vec{B}\right|^2} - 2\,\vec{\kappa}\cdot\vec{\xi}\right), \quad \vec{\kappa} = \frac{\nabla\left(2p + \left|\vec{B}\right|^2\right)}{2\left|\vec{B}\right|^2} - \frac{\vec{B}}{\left|\vec{B}\right|^4}\left(\vec{B}\cdot\nabla\right)\frac{\left|\vec{B}\right|^2}{2}. \tag{15}$$

This system describes the coupled poloidal shear Alfvén and magnetosonic modes (the first and last equations) and the toroidal shear Alfvén modes (the second equation).

In terms of the dimensionless variables

$$\omega \to \omega/\omega_A = \Omega, \quad \left|\vec{B}\right| \to \left|\vec{B}\right|\Big/B_0, \quad \psi \to \psi/\psi_0, \quad p \to p/B_0^2, \tag{16}$$

where

$$B_0 = M/L^3, \quad \psi_0 = M/L, \quad \omega_A = B_0/(L\sqrt{\rho}),$$

and with help of the relations

$$\left(\vec{B}\cdot\nabla\right) = \frac{M}{L^4\cos^7\vartheta}\,\frac{\partial}{\partial\vartheta}, \quad \left|\nabla\psi\right| = \frac{M\sqrt{a(\vartheta)}}{L^2\cos^2\vartheta}, \quad a(\vartheta) = 1 + 3\sin^2\vartheta,$$

which describe the step of going to the derivatives along the magnetic field line, the Eqs. (14) can be presented in the dimensionless form [2]

$$\Omega^2\xi + \frac{a(\vartheta)}{\cos^{13}\vartheta}\,\frac{\partial}{\partial\vartheta}\left[\frac{1}{a(\vartheta)\cos\vartheta}\,\frac{\partial\xi}{\partial\vartheta}\right] - \frac{4}{a(\vartheta)\cos^4\vartheta}\left(T_0 - \frac{\alpha\beta}{\gamma}\,\xi\right) = 0,$$

$$\Omega^2\eta + \frac{1}{\cos^{13}\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\frac{1}{\cos\vartheta}\,\frac{\partial\eta}{\partial\vartheta}\right) = 0, \quad \Omega^2\tau + \frac{1}{\cos^7\vartheta}\,\frac{\partial T_0}{\partial\vartheta} = 0, \tag{17}$$

where

$$T_0 = \frac{\beta\left|\vec{B}\right|^2}{\beta + \left|\vec{B}\right|^2}\left[\frac{1}{\cos^7\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\frac{\tau\,\cos^{12}\vartheta}{a(\vartheta)}\right) - \frac{4\cos^2\vartheta}{a^2(\vartheta)}\,\xi\right], \quad \alpha = -\frac{L}{p}\,\frac{dp}{dL}, \quad \beta = \frac{\gamma p}{B_0^2}. \tag{18}$$

The Eqs. (17) together with the first kind boundary conditions (Dirichlet) corresponding to the fixed magnetic line ends at the perfectly conducting ionosphere fully describe the ballooning modes eigenfrequency spectrum.

STABILITY BOUNDARY

The spectrum of Eqs. (17) for case of "cold" plasma ($\beta = 0$) was analysed previously [3]–[5]. Now we consider the stability of ballooning mode ($\beta > 0$), which can come into existence first of all in the radiation belts. The second of the Eqs. (17), for the perturbation $\eta(\vartheta)$, under the first kind boundary conditions, has a positive spectrum and can not introduce any instability. Therefore, for the needs of stability problem solving, it is necessary to investigate the eigenvalue problem for the other Eqs. (17), namely for perturbations $\xi(\vartheta)$ and $\tau(\vartheta)$, with the same boundary conditions. The stability boundary is determined by the equality $\Omega^2_{min} = 0$, where Ω^2_{min} is the lowest eigenvalue. The equation $\Omega^2_{min}(\alpha, \beta) = 0$ was solved numerically. Stability region for these perturbations in (α, β) plane is shown in Fig. 1. The set of curves for various geomagnetic latitude values ϑ_0 varying from 5° (upper right curve) over 15°, 30°, 45° till 72° (lower left one) is presented. The qualitative estimation $\alpha\,\beta\,\sin^2\vartheta_0 \sim 1$ for these curves takes place. It should be noted that all the eigenfunctions obtained are strongly ϑ-dependent, so the flute eigenmodes are absent in our case of the Dirichlet boundary conditions.

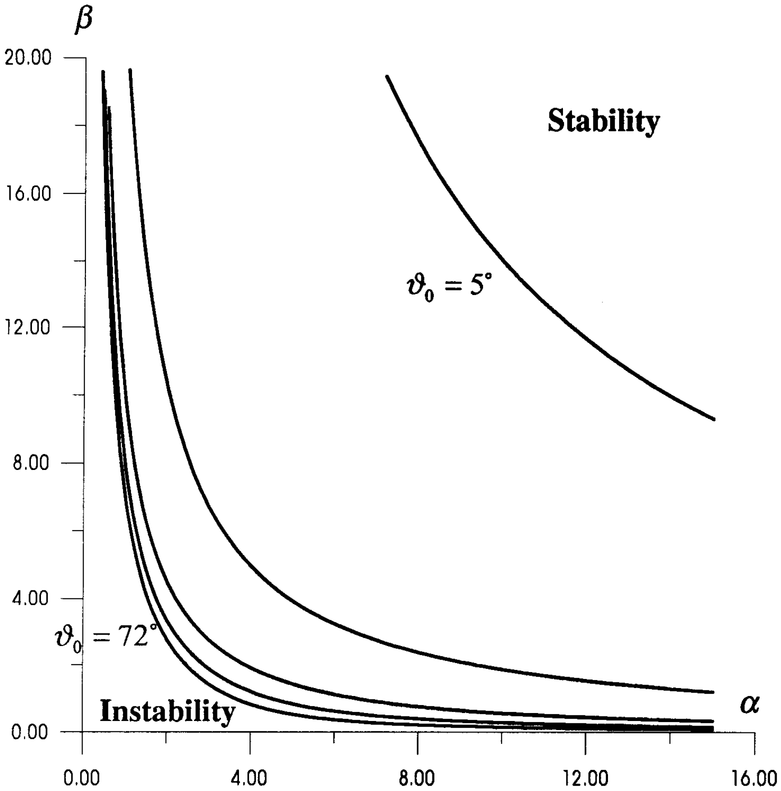

Figure 1: Stability boundaries in the (α, β) plane for different values of geomagnetic latitude ϑ_0 indicating field line ends on the ionosphere

THE MAIN RESULTS

– The system of Eqs. (5), which describes the pressure driven waves in the arbitrary magnetic field, was obtained.

– These equations are reduced to a set of scalar partial differential Eqs. (17)–(18) for ballooning modes in the dipole model of the geomagnetic field.

– The stability region of ballooning modes is obtained.

Acknowledgements. This work was partially supported by the theme 0197U015786 of the National Academy of Sciences of Ukraine and Ministry for Education and Science of Ukraine.

REFERENCES

[1] *Pustovitov V. D., Shafranov V. D.* Reviews of Plasma Physics / Ed. B. B. Kadomtsev.–New York: Consultants Bureau, 1990.–**15**.–P. 163.

[2] *Burdo O. S., Cheremnykh O. K., Verkhoglyadova O. P.* // Izvestiya Akad. Nauk. Ser. Fizicheskaya.–2000.–**64**, N 9b.–P. 1897. (in Russian).

[3] *Burdo O. S., Cheremnykh O. K., Verkhoglyadova O. P.* // Proc. Intern. Symp. "From solar corona through interplanetary space, into Earth's magnetosphere and ionosphere: Interball, ISTP satellites, and ground-based observation".–Kyiv, 2000.–P. 97.

[4] *Chen L., Hasegava A.* // J. Geophys. Res.–1974.–**79**.–P. 1024.

[5] *Cheng C. Z., Chang T. C., Lin C. A., Tsai W. H.* // J. Geophys. Res.–1993.-**18**.–P. 11339.