ESTIMATION OF RANDOM ERRORS OF PHOTOMETRIC METHOD FOR PLANET RELIEF DETERMINATION

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The effect of detection noise on accuracy of the surface relief determination by surface images obtained in the optical or radar ranges has been investigated. The method is based on the brightness dependence on the local normal to the surface. In the frame of proposed technique, statistical estimation of the relief is shown to be not biased. Expressions for variance and spectral density of the error of relief determination have been obtained.

The method of determination of the surface relief according to a series of its optical or radio-images was developed in the work [1]. In [2] it was shown that this method can be applied to determine the surface relief according to its radio-brightness measured by means of the radar with a synthetic aperture (SAR). In [3] the experimental investigation of errors of this method was carried out by computer simulation. These researches have confirmed the efficiency of this method and the acceptable magnitude of its errors. In spite of this it is desirable to study theoretically the errors of this method and to receive analytic expressions for them.

The errors arising when determining the relief by the photometric method described in [1, 2] have a dual origin. Firstly, the signal reception is accompanied by noise of different nature. It causes a random error in the issue; and the higher is the noise level, the larger is the magnitude of error. Secondly, in [1, 2] the equation determining the optimum filter is solved only approximately that finally causes a systematic error, and the steeper surface slopes and the more variations of its parameters are, the larger is the magnitude of error. In the given work we investigate the random error of the photometric method. And in order to avoid bulk calculations, a simplified problem of determination of only the surface relief without determination of its photometric parameters, which are considered as known and constant over all the surface under investigation, is solved.

For examination we shall take, as introduced in [1, 2], the following assumptions. The required relief $H(x,y)$ (*i.e.*, the function describing the dependence of the height H above a mean plane on the Cartesian x, y coordinates) represents the realization of a stationary Gaussian process with the spectral density $1/\alpha(\mathbf{k})$, where $\mathbf{k}$ is two-dimensional vector of spatial frequency. The angle, at which it is possible to see the investigated place from the point of observation, is rather small, so that the direction on the observer can be approximately considered the same in all points of this place. The brightness of the surface element in the j-th image depends on the orientation of the normal $\mathbf{n}$ to the element by the law

$$I_j = F_j\left(\mathbf{n}\right), \tag{1}$$

where F_j is an known function describing the properties of the scattering surface. Assuming that the surface inclinations are small, it is possible to expand $F_j\left(\mathbf{n}\right)$ into a Taylor series in terms of $\mathbf{n}$ and the average normal to the surface $\mathbf{n}_0$. Keeping the first expansion term one can write:

$$I_j = I_{0j} + \mathbf{c}_j \cdot \left(\mathbf{n} - \mathbf{n}_0\right). \tag{2}$$

When $|\mathbf{n} - \mathbf{n}_0|$ is small, this relation within the accuracy of second order infinitesimals can be written as

$$I_j = I_{0j} + \mathbf{c}_j \cdot \mathbf{t}\left(x,y\right), \tag{3}$$

where

$$\mathbf{t}\left(x,y\right) = \nabla H\left(x,y\right) \tag{4}$$

is the two-dimensional vector of the surface inclination, ∇ is the two-dimensional gradient, and $\mathbf{c}_j$ is the two-dimensional vector constant determined by properties of the surface and the actual geometry. Each realization of the relief $H(x,y)$ is characterized by its own value of the prior density

$$\rho_h(H) \cong exp\left(-\int\limits_{-\infty}^{\infty} \tilde{H}^*(\mathbf{k})\ \alpha(\mathbf{k})\ \tilde{H}(\mathbf{k})\ d^2\mathbf{k}\right) \tag{5}$$

(note that the normalizing factor does not play any important role, because it does not enter into the final result). Similarly for noise

$$\rho_n(N) \cong exp\left(-\int\limits_{-\infty}^{\infty} \tilde{N}^*(\mathbf{k})\ \beta(\mathbf{k})\ \tilde{N}(\mathbf{k})\ d^2\mathbf{k}\right). \tag{6}$$

Here $\tilde{H}(\mathbf{k})$ and $\tilde{N}(\mathbf{k})$ are Fourier-images of the relief and noise accordingly, $\beta(\mathbf{k})$ is the value inverse to the noise spectral density, the asterisk means complex conjugation.

The images received in an experiment and described by the brightness distribution $J_j(x,y)$ are related with $I_j(x,y)$ by the relation

$$J_j(x,y) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} g_j(x-x', y-y')\ I_j(x',y')\ dx'\,dy' + N_j(x,y), \tag{7}$$

where $g_j(x-x', y-y')$ is the kernel describing the j-th image diffusion caused by phase distortions in the propagation medium and other disturbing factors, $N_j(x,y)$ is an arbitrary realization of the stationary additive Gaussian noise. This relation is more convenient *ad referendum* to write in the frequency representation expanding both parts into a Fourier series

$$\tilde{J}_j(\mathbf{k}) = \tilde{g_j}(\mathbf{k})\ \tilde{I}_j(\mathbf{k}) + \tilde{N}_j(\mathbf{k}). \tag{8}$$

Let us have m of such images received from different points of the trajectory of an aircraft. It is required to find the most probable relief.

The expression for *a posteriori* density of the relief $H(x,y)$ is generated by the Bayes' formula [4] taking into account (12), (14), (15). Later it will be more convenient to use its logarithm

$$L(H) = \int \tilde{H}^*(\mathbf{k})\ \alpha(\mathbf{k})\ \tilde{H}(\mathbf{k})\ d^2\mathbf{k} +$$

$$+\sum_j \int \left[\tilde{J}_j(\mathbf{k}) - i\,\tilde{g_j}(\mathbf{k})\ \mathbf{c}_j\cdot\mathbf{k}\ \tilde{H}^*(\mathbf{k})\right]^* \beta(\mathbf{k}) \left[\tilde{J}_j(\mathbf{k}) - i\,\tilde{g_j}(\mathbf{k})\ \mathbf{c}_j\cdot\mathbf{k}\ \tilde{H}^*(\mathbf{k})\right] d^2\mathbf{k}. \tag{9}$$

The maximum of this functional determines the most probable relief $H_m(x,y)$ for the given set of original images $J_j(x,y)$ $(i = 1, \ldots, m)$.

When equating to zero the derivative of this expression with respect to $\tilde{H}(\mathbf{k})$, one shall receive the equation for the most probable relief

$$\left[\alpha(\mathbf{k}) + \sum_j (\mathbf{k}\,\mathbf{c}_j)^2\ \tilde{g_j}^*(\mathbf{k})\ \tilde{g_j}(\mathbf{k})\ \beta(\mathbf{k})\right] \tilde{H}(\mathbf{k}) = i\sum_j \mathbf{k}\,\mathbf{c}_j\ \tilde{g_j}^*(\mathbf{k})\,\beta(\mathbf{k})\,\tilde{J}_j(\mathbf{k}), \tag{10}$$

its solution is

$$\tilde{H}_m(\mathbf{k}) = \left[\alpha(\mathbf{k}) + \sum_j (\mathbf{k}\,\mathbf{c}_j)^2\ \tilde{g_j}^*(\mathbf{k})\ \tilde{g_j}(\mathbf{k})\ \beta(\mathbf{k})\right]^{-1} i\sum_j \mathbf{k}\,\mathbf{c}_j\ \tilde{g_j}^*(\mathbf{k})\,\beta(\mathbf{k})\,\tilde{J}_j(\mathbf{k}). \tag{11}$$

This expression determines the optimum filter, which translates whole set of images $J_j(x,y)$ received as a result of the experiment into the most probable relief $H_m(x,y)$. It is a particular case of the result received in [1, 2], and comes out from it if all its photometric parameters are considered as constants.

Let the true relief be described by the function $H_0(x,y)$. Then, as a result of the experiment, we shall get according to (7), (3), (4) a set of images

$$J_j(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(x-x', y-y')\ \mathbf{c}_j\,\nabla H_0(x',y')\,dx'\,dy' + N_j(x,y)\,, \tag{12}$$

where $N_j(x,y)$ are independent realizations of noise during recording. Processing it with the optimum filter (11), we shall obtain the most probable relief for the given set of images $J_j(x,y)$, which in the frequency representation can be written as follows

$$\tilde{H}_m(\mathbf{k}) = P(\mathbf{k})\ \tilde{H}_0(\mathbf{k}) + \sum_j Q_j(\mathbf{k})\ \tilde{N}_j(\mathbf{k})\,, \tag{13}$$

where

$$P(\mathbf{k}) = \frac{\sum\limits_j (\mathbf{k}\,\mathbf{c}_j)^2\, \tilde{g_j}^*(\mathbf{k})\ \tilde{g_j}(\mathbf{k})}{\dfrac{\alpha(\mathbf{k})}{\beta(\mathbf{k})} + \sum\limits_j (\mathbf{k}\,\mathbf{c}_j)^2\, \tilde{g_j}^*(\mathbf{k})\ \tilde{g_j}(\mathbf{k})}, \tag{14}$$

$$Q(\mathbf{k}) = \frac{i\,\mathbf{k}\,\mathbf{c}_j\tilde{g_j}^*(\mathbf{k})}{\dfrac{\alpha(\mathbf{k})}{\beta(\mathbf{k})} + \sum\limits_j (\mathbf{k}\,\mathbf{c}_j)^2\, \tilde{g_j}^*(\mathbf{k})\ \tilde{g_j}(\mathbf{k})}. \tag{15}$$

Thus, we have received the relationship between the true relief and the statistical estimation of the most probable relief, in which unknown realizations of noise $N_j(x,y)$ are present, and because no one of these realizations $\tilde{H}_m(\mathbf{k})$ is equal to $\tilde{H}_0(\mathbf{k})$. The difference

$$\delta(x,y) = H_m(x,y) - H_0(x,y) \tag{16}$$

is the error of determination of the relief, the random nature of which is caused by the influence of the presence into (12) realizations of noise $N_j(x,y)$ and by random nature of the required relief (which also contributes to the error beyond the filter passband). Writing down (16) as a frequency representation and substituting (3) and (12) in it, we shall get

$$\delta(\mathbf{k}) = (P(\mathbf{k}) - 1)\ \tilde{H}_0(\mathbf{k}) + \sum_j Q_j(\mathbf{k})\ \tilde{N}_j(\mathbf{k}). \tag{17}$$

From this expression one can see that two components of essentially different nature make a random error of the relief definition. The addend linearly depends on noise realizations and as a matter of fact it represents the noise, which runs through the filter and forms the relief from original images. The augend comes from the signal itself, depends linearly on it and is related to the smoothing nature of the forming filter. Both components depend on the frequency nature of the filter, and so on statistical properties of the relief and noise. However, the augend does not depend on specific realizations of noise $N_j(x,y)$, and the addend is independent on the true relief $H_0(x,y)$.

As $H_0(x,y)$ and $N_j(x,y)$ are stationary Gaussian processes, owing to (17) this concerns $\delta(x,y)$ as well. Consequently, values of $\delta(\mathbf{k})$ for different $\mathbf{k}$ are not correlated with each other. Therefore, statistical properties of the error $\delta(x,y)$ are completely characterized by the function

$$\tilde{D}(\mathbf{k}) = \overline{\tilde{\delta}^*(\mathbf{k})\ \tilde{\delta}(\mathbf{k})}, \tag{18}$$

where the line means the ensemble averaging of signal and noise realizations. Substituting $\delta(\mathbf{k})$ from (17) in this expression and considering that noise realizations $\tilde{N}_j(\mathbf{k})$ are not correlated with the relief and with each other, we receive

$$\tilde{D}(\mathbf{k}) = (P(\mathbf{k}) - 1)^2\ \overline{\tilde{H}_0^*(\mathbf{k})\,\tilde{H}_0(\mathbf{k})}\ + \sum_j Q_j^*(\mathbf{k})\,Q_j(\mathbf{k})\ \overline{\tilde{N}_j^*(\mathbf{k})\,\tilde{N}_j(\mathbf{k})}. \tag{19}$$

This expression represents the error spectral density $\delta(x,y)$. Substituting here $1/\alpha(\mathbf{k})$ and $1/\beta(\mathbf{k})$ instead of the average in the addend and the augend accordingly with the account of (14) and (15) we shall obtain

$$\tilde{D}(\mathbf{k}) = \frac{1}{\alpha(\mathbf{k}) + \beta(\mathbf{k}) \sum_j (\mathbf{k}\,\mathbf{c}_j)^2 \tilde{g_j}^*(\mathbf{k})\, \tilde{g_j}(\mathbf{k})}. \tag{20}$$

The function $D(x, y)$, the Fourier-image of which is $\tilde{D}(\mathbf{k})$, represents the autocorrelation function of this error. The existence of this function is provided by that for the smooth relief the augend in the denominator (20) tends to infinity when $\mathbf{k}$ approaches infinity. As the addend in the denominator usually decreases quickly at infinity owing to the limited spatial passband of the section (what is expressed by the fast decrease of $\tilde{g}(\mathbf{k})$ with the increase of the module $\mathbf{k}$, the function $\tilde{D}(\mathbf{k})$ tends to zero at infinity according to the same law as the spectral density of the relief (the tending of the latter to zero at infinity is supposed because of the true relief regularity). It follows, in particular, that $D(x, y)$ has the same degree of smoothness (has the same quantity of derivative) as the autocorrelation function of the relief.

When analyzing (19) it is easy to see that, as one would expect, on a set of arbitrary frequency characteristics of the filter, which forms the relief from original images, the error dispersion $\tilde{D}(\mathbf{k})$ reaches minimum for all $\mathbf{k}$, if this filter is determined by the equation (11). Thus, the filter (11) being optimal to the effect that gives the most probable relief is the optimum at the same time to the effect that provides the minor root-mean-square error of determination of each Fourier-component of the relief.

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