TEMPERATURE DISTRIBUTION IN A SEMI–INFINITE ATMOSPHERE SUBJECTED TO COSINE COLLIMATED RADIATION

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Accurate numerical solutions are presented for the radiation field in a semi-infinite, two-dimensional, plane-parallel, absorbing-emitting but non-scattering gray atmosphere subjected to cosine-varying collimated incident boundary radiation. We approximate the kernel of the integral equation for the emissive power by a sum of exponents. After this approximation the integral equation can be solved exactly. The solution contains the well-known Ambartsumian-Chandrasekhar $H-$function. This approach allowed us to find the accurate values for the emissive power and the radiative flux at arbitrary optical depths in the atmosphere. The calculations have shown that the radiative flux may have a maximum at certain values of the spatial frequency in the atmosphere and that the region, where the emissive power reaches a constant value, may lie very deep in the atmosphere.

INTRODUCTION

An one-dimensional model for radiative transfer in different media has been extensively studied. Many problems in one-dimensional transfer allow rigorous mathematical solutions which may serve as benchmarks for more complicated cases or as first approximations to two-dimensional problems.

However, there are many problems in astrophysics where the results of the one-dimensional model of radiative transfer are not accurate enough and we have to apply models of two- or three-dimensional radiative transfer. This enormously complicates the solution and thus only a few exact studies exist and they deal mostly with scattering of a narrow pencil of radiation incident on a scattering medium (*cf.* the references in [1]).

There is a group of two-dimensional problems for which an exact solution can be found. One of these problems is connected with non-scattering media with the cosine varying incident collimated radiation. In this case the two-dimensional problem can be reduced to one-dimensional integral equation as shown by Breig and Crosbie [2]. However, their approach allowed to determine only the external radiation field.

In this report we generalize the results of Breig and Crosbie by applying the method of approximating the kernel of the integral equation for the Sobolev resolvent function (respective Green function) by a series of exponents. The resulting approximate equation has an exact solution which is also represented by a series of exponents. This allows us to define the auxiliary functions g and h through the resolvent function Φ and thus define all the relevant functions.

In one-dimensional media the described approach gave very accurate results [3]. It appeared that for the problem on hand we were able to find both the external and internal radiation field.

The full-scale presentation of this problem is given in [1] and for the optically finite atmosphere in [4].

SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

We are looking for the emissive power in a homogeneous non-scattering plane-parallel two-dimensional gray atmosphere which is in local thermodynamic equilibrium. The radiative transfer in such an atmosphere is described by the following equation

$$\cos\theta\frac{\partial I}{\partial\tau_z} + \sin\theta\sin\phi\frac{\partial I}{\partial\tau_y} + I = \frac{\sigma}{\pi}T^4, \tag{1}$$

where I is the intensity; θ is the polar angle measured from the inward normal to the atmosphere; ϕ is azimuthal angle measured from the τ_x-axis; σ is the Stefan-Boltzmann constant; T is the temperature in the atmosphere; σT^4 is the emissive power. The optical depth τ_z is measured downward from the boundary of the atmosphere and together with τ_x and τ_y they form a right-hand rectangular co-ordinate system. We require that the energy is transferred only by radiation, *i.e.*, there is no heat conduction or convection in the atmosphere.

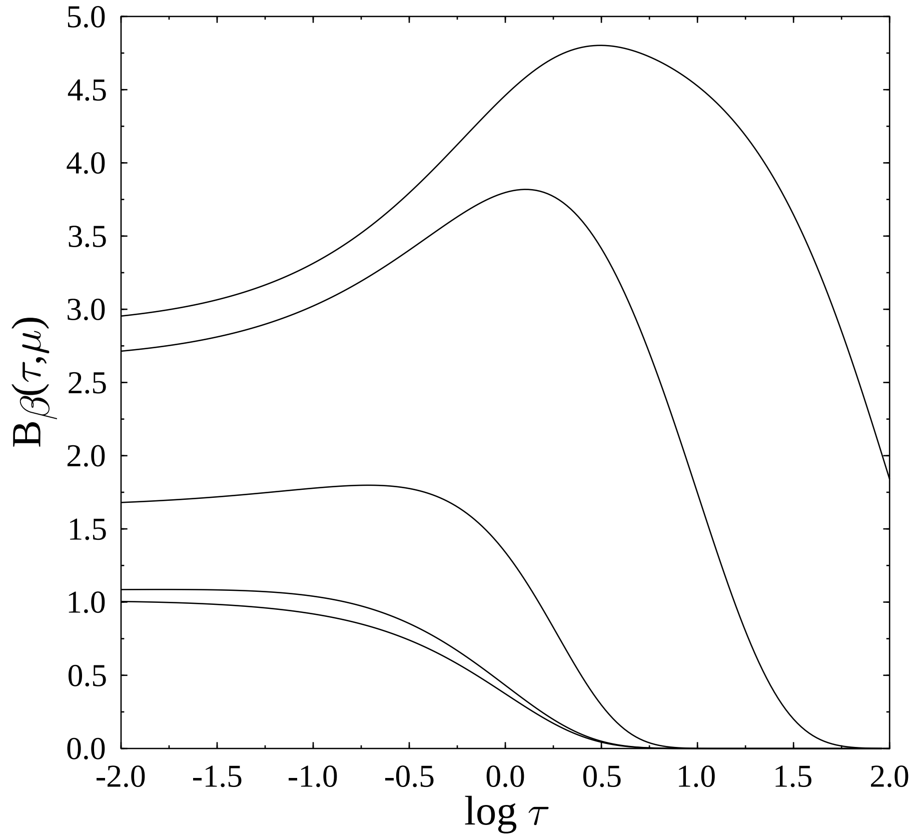

Figure 1: The dimensionless emission power as a function of the optical coordinate τ_z for different values of spatial frequency β while the angle of incidence is zero. Starting from above: $\beta = -2, -1, 0, 1, 2$

Taking this fact into account we obtain the equation for the emissive power

$$4\sigma T^4(\tau_y, \tau_z) = \int_{2\pi} I_0^+(\tau_y^+) \exp(-\tau_z/\mu) \mathrm{d}\omega + \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_0^\infty \sigma T^4(\tau_y', \tau_z') \exp(-|\tau_z - \tau_z'|/\mu) \mathrm{d}\tau_z' \mathrm{d}\mu/\mu \mathrm{d}\phi, \qquad (2)$$

where $\tau_y^+ = \tau_y - \tau_z \tan\theta \sin\phi$, $\tau_y' = \tau_y + (\tau_z' - \tau_z) \tan\theta \sin\phi$, $\mu = \cos\theta$ and I_0^+ is the intensity incident on the boundary of the atmosphere [2]. According to our assumption the incident intensity may be expressed as

$$I_0^+(\tau_y^+) = I_0 \left[1 + \epsilon \exp(i\beta\tau_y^+)\right] \delta(\mu - \mu_0)\delta(\phi), \qquad (3)$$

where I_0 is a constant, $(\mu_0 = \cos\theta_0, \phi)$ defines the direction of the incident collimated radiation, ϵ is the amplitude of the cosine wave and δ is the Dirac delta function. Boundary condition (3) means that the top of the atmosphere is strip-wise illuminated by a parallel beam at an angle θ_0 while the strips are parallel to the x-axis and their widths are defined by the spatial frequency β as π/β in units of optical length τ_y^+. The illumination in the direction parallel to the y-axis varies according to the cosine law. Next we apply the concept of separation of variables to Eq. (2) by assuming that

$$\sigma T^4(\tau_y, \tau_z) = \frac{1}{4} I_0 \left[B_{\beta=0}(\tau, \mu_0) + \epsilon B_\beta(\tau, \mu_0) \exp(i\beta\tau_y)\right], \qquad (4)$$

where B_β is the dimensionless emissive power and $\tau = \tau_z$. Using Eq. (4) in Eq. (2) gives us a simple integral equation for B_β in the form

$$B_\beta(\tau, \mu_0) = \exp(-\tau/\mu_0) + \frac{1}{2} \int_0^\infty \mathcal{E}_1(\tau - \tau') B_\beta(\tau', \mu_0) \mathrm{d}\tau', \qquad (5)$$

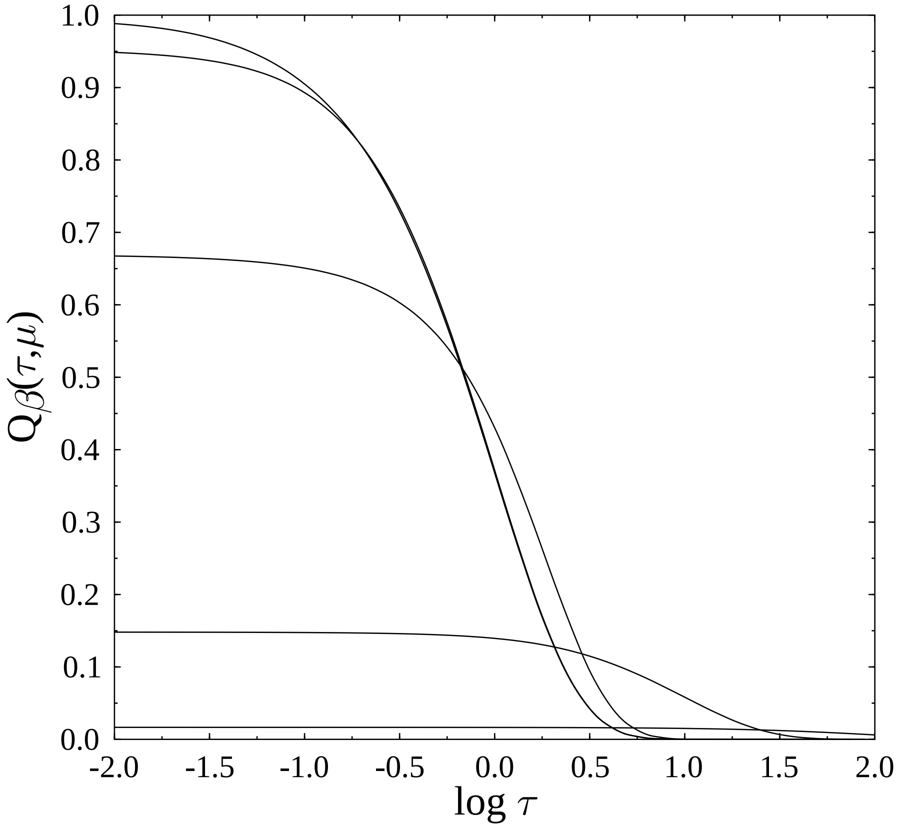

Figure 2: The dimensionless flux as a function of the optical coordinate τ_z for different values of spatial frequency β while the angle of incidence is zero. Starting from above: $\beta = -2, -1, 0, 1, 2$

where the generalized exponential integral $\mathcal{E}_1$ is defined as [2]

$$\mathcal{E}_1(\tau, \beta) = \int_1^\infty \exp(-|\tau| \sqrt{u^2 + \beta^2}) \frac{\mathrm{d}u}{\sqrt{u^2 + \beta^2}}. \tag{6}$$

Sobolev has shown that the non-singular part of the Green function for Eq. (5) – often called the Sobolev resolvent function Φ satisfies the following equation [5]

$$\Phi_\beta(\tau) = \frac{1}{2}\mathcal{E}_1(\tau, \beta) + \frac{1}{2}\int_0^\infty \mathcal{E}_1(\tau - \tau')\Phi_\beta(\tau')\mathrm{d}\tau'. \tag{7}$$

Next we introduce two functions $h(\tau, \mu)$ and $g(\tau, \mu)$ as follows [3]

$$h_\beta(\tau, \mu) = 1 + \int_\tau^\infty \Phi_\beta(t) \exp(-(t - \tau)/\mu)\mathrm{d}t, \tag{8}$$

$$g_\beta(\tau, \mu) = \exp(-\tau/\mu) + \int_0^\tau \Phi_\beta(t) \exp(-(\tau - t)/\mu)\mathrm{d}t. \tag{9}$$

Sobolev [5] has shown that the solution of Eq. (5) may be written in our notation in the form

$$B_\beta(\tau, \mu_0) = B_\beta(0, \mu_0) g_\beta(\tau, \mu_0). \tag{10}$$

Formally, this completes the solution of the problem.

In order to find the functions $h_\beta(\tau, \mu_0)$ and $g_\beta(\tau, \mu)$ at an arbitrary optical depth we have to solve Eq. (7). This can be easily done if we approximate the kernel of Eq. (7) by a sum of exponents. After that the integral equation can be solved exactly and we may find the functions $h(\tau,\mu)$ and $g(\tau,\mu)$ by simple integration. This procedure is described in detail in [3].

NUMERICAL RESULTS

Due to the facts that for large β the characteristic function of the problem increases rapidly towards the end of the domain of its definition and that the generalized exponential integral has a logarithmic singularity at $\tau \to 0$ we have to be careful in using the Gaussian rule in approximating the kernel of Eq. (7). For better accuracy we divided the integration range into four subintervals and in each subinterval we used the Gaussian rule with $N/4 = 21$ points. This scheme ensured at least five significant digits both for the dimensionless flux and the emissive power in the region of $0 \le \beta \le 10000$.

Fig. 1 shows the behaviour of the dimensionless emissive power for $\mu = 1.0$. According to calculations for small angles of incidence the dimensionless emissive power decreases monotonically for all the values of β. This does not take place in the case of perpendicular incidence where such a monotonous decrease is present only for $\log\beta \ge 0.5$. For smaller values of β the dimensionless emissive power has a maximum at a definite τ.

The influence of the spatial frequency β and the optical depth τ (optical coordinate in z-direction) on the dimensionless flux of energy $Q_\beta(\tau,\mu=1)$ is illustrated in Fig. 2. The dimensionless flux decreases from its maximum value (which is equal to μ) at large values of spatial frequency and small values of optical depth towards smaller values of spatial frequency and larger values of optical depth but curiously enough there appears a maximum in the $Q_\beta - \log\tau$ plane in the region where $\log\tau \ge -0.75$.

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